Towards a Foundational Principle for Quantum Mechanics

The wind is not moving, the flag is not moving.
Mind is moving [1]

Terence J. Nelson
New Providence, NJ USA (terry@tnelson.com)

This paper opens a new approach to quantum mechanics starting from the proposition that particles and fields interact discretely through quantized exchanges of momentum that occur at predictable rates but at otherwise unpredictable times. It leads to imaginary rates and complex probability densities. It is argued that the probability that a detector fires should be proportional to the absolute magnitude of the predicted complex probability density and this is confirmed by application to the double-slit problem. The calculation is performed by a path-integral method using familiar mathematics but no wave function is involved, only a complex probability density.

Introduction

When I first heard about Heisenberg's uncertainty principle, I supposed that the statistical nature of the predictions of quantum mechanics might emerge from the unpredictable timing of individual exchanges of quanta between fields and material particles. However, the academic development didn't take this path. What ensued was best characterized by Richard Feynman [2] when he said "we cannot make the mystery go away … we will just tell you how it works."

After thinking about it for a long time, I may have found a simple way forward along the path I had originally anticipated. The first part of the foundational principal (notably missing from conventional quantum theory as pointed out by Stephen Boughn [3]) that I want to propose is:

a) **Particles and fields interact discretely through quantized exchanges of momentum that occur at predictable rates but at otherwise unpredictable times.**

This should probably be obvious, because one could only discover the timing of a first exchange with additional interactions that would compound the uncertainty. However, starting from this point gets us deeper into the strangeness of quantum mechanics by suggesting imaginary exchange rates and complex probability densities.

Now suppose we place a detector at $\boldsymbol{x}$. It should be clear that a typical particle detector's operation shouldn't depend on the phase of the complex probability density at $\boldsymbol{x}$ because phase is relative and no input is either accepted by the detector nor available for a reference. Therefore it is proposed that the second part of the foundational principle should be

> b) **The probability of a particle detector at position *x* firing is proportional to the absolute magnitude of the complex probability density at *x*.**

Evidently, the phase of the complex probability density is lost when the detector clicks, but instead of saying that the wave function has collapsed, we can say that the details of the interaction with the detector are not and probably cannot be accounted for consistently. That is, detection is inherently non-unitary.

1. Building on the Foundation

The force acting on a particle is the negative of the gradient of the potential energy. When the potential, say $V(\boldsymbol{x})$, is expressed as a Fourier integral, the integrand differs from that of the potential by a factor of $-i\boldsymbol{k}$,

$$\begin{aligned}\boldsymbol{F} &= -\nabla V(\boldsymbol{x}) \\ &= -i(1/\hbar)\int d^3k\,(\hbar\boldsymbol{k})V(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\end{aligned}$$

On the other hand, $\hbar\boldsymbol{k}$ is the momentum per quanta exchanged with the field. What is left after dividing by $\hbar$, should be the integral of the rate of exchange of such quanta in d^3k. We conclude that total rate of exchange of quanta at all wave vectors at position $\boldsymbol{x}$ is $-i(1/\hbar)V(\boldsymbol{x})$. Formally applying Poisson statistics and exponentiating the negative of the expected number of exchanges in a time dt then gives a "probability"

$$P_a = e^{i(1/\hbar)V(\boldsymbol{x})dt}$$

that no exchange between a particle at $\boldsymbol{x}$ and the field occurs in time dt. (The subscript just indicates that we are not done defining the total probability yet.) Of course, this probability is not necessarily a positive real number, so its application to experimental data is not any more intuitive than in conventional quantum mechanics, where Born's rule must be counted as an independent hypothesis. On the other hand, our complex probability provides a mathematical object to which we can apply superposition, which is a well-established feature of matter at the quantum scale.

It appears that the assumption that quanta are transferred at definite, albeit unknowable, positions and times is what makes superposition inescapable and may be the source of quantum “weirdness” generally. Ultimately, we should concede that, as thinking beings, we have access only to durable records of measurements. In particular, time itself is a classical construct that we use to relate changes in physical systems of interest with respect to some particularly stable system serving as a clock. Of course, changes in the clock reading at its finest possible scale are also quantum jumps. Therefore we can speculate that a more fundamental approach should be able to account for perceived reality by referring only to exchanges of quanta between various systems.

Lacking a more general way forward, we will proceed as though time is real and continuous, but we still need a term that represents the effects of inertia. If the potential energy is a measure of the average rate at which quanta are exchanged with the field, the mechanical energy could be similarly related to interactions with whatever it is (presumably the Higgs field) that gives particles their mass and inertia. Adding the total mechanical energy *T* to the potential *V* then gives:

$$P_b = e^{i(1/\hbar)\left(V(\boldsymbol{x})+T(p)\right)dt}$$

for the probability of no exchange in time *dt*. Here ***p*** is the momentum of the particle before any interaction and *T*(*p*) is the total (kinetic + rest) mechanical energy.

Now consider a particle starting from some location $\boldsymbol{x}_0$ at time t_0 with momentum $\boldsymbol{p}_0$. The complex probability density that it will arrive at some other position $\boldsymbol{x}_1$ at time t_1 without having an exchange should be

$$\rho_c(\boldsymbol{x}_1, t_1; \boldsymbol{x}_0, t_0, \boldsymbol{v}_0) = e^{i(1/\hbar)\int_{t_0}^{t_1} dt'(V(\boldsymbol{x}')+T(\boldsymbol{p_0}))} \delta^3\left(\boldsymbol{x}_1 - \boldsymbol{x}_0 - \boldsymbol{v}_0\left(t_1 - t_0\right)\right)$$

because multiplying the probabilities for each increment of time results in adding the differential phase terms in the exponent. They add up to a path integral of the sum of the potential and mechanical energies. Of course the initial momentum and position can't be known simultaneously, because instruments that might be used to determine them are also unpredictable at the quantum level. Nevertheless, the classical metaphysical principle, that a particle has a definite position and momentum at each instant of time, is not abandoned in the present development.

Continuing with the formal application of Poisson statistics suggests that the differential probability of exchange of a single quantum with wave vector $\boldsymbol{k}_1$ in d^3k_1 during a time dt_1 is

$$d^4P_d = -i(1/\hbar)dt_1 d^3k_1 V(\boldsymbol{k}_1)e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1 + i(1/\hbar)(V(\boldsymbol{x}_1)+T(\boldsymbol{p}_1))dt_1}$$

After picking up momentum $\hbar\boldsymbol{k}_1$, the particle will be moving at constant velocity on a new trajectory until the next quantum is exchanged. The amplitude for arriving at some final position ***x*** at time *t* can be expressed as an integral along the initial trajectory of a product of three terms. The first term is the probability for not interacting along the initial leg, which is given by P_c if we say the first quantum is exchanged at t_1. The second term is the amplitude for the first exchange, which modifies the trajectory that the particle will follow to the second exchange:

$$\boldsymbol{p}_1 = \boldsymbol{p}_0 + \hbar \boldsymbol{k}_1,$$
$$\frac{\boldsymbol{v}_1}{c} = \frac{\boldsymbol{p}_0 + \hbar \boldsymbol{k}_1}{\left(\left|\boldsymbol{p}_0 + \hbar \boldsymbol{k}_1\right|^2 + m^2 c^2\right)^{1/2}}$$
$$\boldsymbol{x}_2 = \boldsymbol{x}_1 + (t_2 - t_1)\boldsymbol{v}_1$$

The third term is the amplitude for arriving at the final position $\boldsymbol{x}$ at time t with the starting position shifted to $\boldsymbol{x}_1$ and t_1. This reasoning suggests an integral equation

$$\rho_e(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) = e^{i(1/\hbar)\int_{t_0}^{t} dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}_0))} \delta^3(x - \boldsymbol{x}_0 - \boldsymbol{v}_0(t-t_0))$$
$$-i(1/\hbar) \int_{-\infty}^{c(t-t_1)\geq|x-\boldsymbol{x}_1|} dt_1 e^{i(1/\hbar)\int_{t_0}^{t_1} dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}_0))} \int d^3k_1 V(\boldsymbol{k}_1) e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1} \rho_e(\boldsymbol{x},t;\boldsymbol{x}_1,t_1,\boldsymbol{v}_1)$$

where it can be noted that $d^3kV(\boldsymbol{k})$ has the dimensions of energy, and the effective integration over the space-time coordinates is restricted by relativistic causality. (Note that the particle can only stay on the initial trajectory until it would have to reach the velocity of light on the final trajectory.)

Now we can rewrite the phase factor that comes from the Fourier transform accurately as

$$i\boldsymbol{k}_1 \cdot \boldsymbol{x}_1 = (i/\hbar)(\boldsymbol{p}_1 - \boldsymbol{p}_0)\cdot \boldsymbol{x}_1$$

which combines with the path integral of the energy on a segment to produce a factor of the form:

$$e^{i(1/\hbar)\left\{\int_{t_0}^{t_1} dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}'))+(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1\right\}} = e^{i(1/\hbar)\boldsymbol{p}_1\cdot\boldsymbol{x}_1} e^{i(1/\hbar)\left\{\int_{t_0}^{t_1} dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}'))-\boldsymbol{p}_0\cdot(\boldsymbol{x}_1-\boldsymbol{x}_0)\right\}} e^{-i(1/\hbar)\boldsymbol{p}_0\cdot\boldsymbol{x}_0}$$
$$= e^{i(1/\hbar)\boldsymbol{p}_1\cdot\boldsymbol{x}_1} e^{i(1/\hbar)\int_{t_0}^{t_1} dt'[V(\boldsymbol{x}')+T(\boldsymbol{p}')-\boldsymbol{p}'\cdot\boldsymbol{v}']} e^{-i(1/\hbar)\boldsymbol{p}_0\cdot\boldsymbol{x}_0}$$

which appears to be related to the Feynman path integral since

$$V(\boldsymbol{x}')+T(\boldsymbol{p}')-\boldsymbol{p}'\cdot\boldsymbol{v}' = L(\boldsymbol{x}',\boldsymbol{p}')$$

is a possible definition for the relativistic Lagrangian. Now, the velocity between t_0 and t_1 is $\boldsymbol{v}_0$, and $\hbar\boldsymbol{k}_1 = \boldsymbol{p}_1 - \boldsymbol{p}_0$, so we can rearrange terms to express the equation in a potentially simpler form as

$$\rho_e(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$

$$= e^{i(1/\hbar)\boldsymbol{p}_0\cdot(\boldsymbol{x}-\boldsymbol{x}_0)} e^{i(1/\hbar)\int_{t_0}^{t} dt'[V(\boldsymbol{x}')+T(\boldsymbol{p}_0)-\boldsymbol{p}_0\cdot\boldsymbol{v}']} \delta^3(x-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0))$$

$$-i(1/\hbar)\int_{-\infty}^{c(t-t_1)\geq|x-\boldsymbol{x}_1|} dt_1 e^{i(1/\hbar)\int_0^1 dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}')-\boldsymbol{p}'\cdot\boldsymbol{v}')} \int d^3k_1 V(\boldsymbol{k}_1) e^{i(1/\hbar)\boldsymbol{p}_0\cdot(\boldsymbol{x}_1-\boldsymbol{x}_0)} e^{i(1/\hbar)(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1} \rho_e(\boldsymbol{x},t;\boldsymbol{x}_1,t_1,\boldsymbol{v}_1)$$

Now, introducing an operator $\boldsymbol{p}$ to be replaced by the momentum on the final segment of the path being considered we can write

$$e^{-i(1/\hbar)(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{p}_0\cdot\boldsymbol{x}_0)}\rho_e(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$

$$= e^{-i(1/\hbar)(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{p}_0\cdot\boldsymbol{x}_0)} e^{i(1/\hbar)\boldsymbol{p}_0\cdot(\boldsymbol{x}-\boldsymbol{x}_0)} e^{i(1/\hbar)\int_{t_0}^{t} dt' L(\boldsymbol{x}',\boldsymbol{v}_0)} \delta^3(x-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0))$$

$$-i(1/\hbar)\int_{-\infty}^{c(t-t_1)\geq|x-\boldsymbol{x}_1|} dt_1 e^{i(1/\hbar)\int_{t_0}^{t_1} dt' L(\boldsymbol{x}',\boldsymbol{v}')} \int d^3k_1 V(\boldsymbol{k}_1) e^{-i(1/\hbar)(\boldsymbol{p}_1\cdot\boldsymbol{x}-\boldsymbol{p}_0\cdot\boldsymbol{x}_0)} e^{i(1/\hbar)\boldsymbol{p}_0\cdot(\boldsymbol{x}_1-\boldsymbol{x}_0)} e^{i(1/\hbar)(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1} \rho_e(\boldsymbol{x},t;\boldsymbol{x}_1,t_1,\boldsymbol{v}_1)$$

The extra phase cancels with $\boldsymbol{p}_0$ acting on the direct path integral leaving

$$e^{-i(1/\hbar)(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{p}_0\cdot\boldsymbol{x}_0)}\rho_e(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) = e^{i(1/\hbar)\int_{t_0}^{t} dt' L(\boldsymbol{x}',\boldsymbol{v}_0)} \delta^3(x-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0))$$

$$-i(1/\hbar)\int_{-\infty}^{c(t-t_1)\geq|x-\boldsymbol{x}_1|} dt_1 e^{i(1/\hbar)\int_{t_0}^{t_1} dt' L(\boldsymbol{x}',\boldsymbol{v}')} \int d^3k_1 V(\boldsymbol{k}_1) e^{-i(1/\hbar)(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{p}_1\cdot\boldsymbol{x}_1)} \rho_e(\boldsymbol{x},t;\boldsymbol{x}_1,t_1,\boldsymbol{v}_1)$$

Thus our proposed integral equation for ρ can be transformed into one for ρ_f defined by

$$\rho_f(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) = e^{-i(1/\hbar)(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{p}_0\cdot\boldsymbol{x}_0)}\rho_e(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$

and satisfying

$$\rho_f(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) = e^{i(1/\hbar)\int_{t_0}^{t} dt' L(\boldsymbol{x}',\boldsymbol{v}_0)} \delta^3(x-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0))$$

$$-i(1/\hbar)\int_{-\infty}^{c(t-t_1)\geq|x-\boldsymbol{x}_1|} dt_1 e^{i(1/\hbar)\int_{t_0}^{t_1} dt' L(\boldsymbol{x}',\boldsymbol{v}')} \int d^3k_1 V(\boldsymbol{k}_1)\rho_f(\boldsymbol{x},t;\boldsymbol{x}_1,t_1,\boldsymbol{v}_1)$$

We propose to adopt ρ_f as our final definition of the complex probability density as it has the correct dimensions and Lorentz transformation properties. Also, when multiple segments are cascaded without interactions, the extra phase factors cancel at internal nodes leaving the final phase increment expected for a plane wave. Accordingly, we will drop the subscript “ f ” hereafter.

2. Perturbation Approach

We next assume that the complex probability for a particle starting at position $\boldsymbol{x}_0$ at time t_0 with velocity $\boldsymbol{v}_0$ to arrive at $\boldsymbol{x}$ at time t can be obtained from a perturbation approach based on the number of quanta exchanged.

$$\rho(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)=\sum_{n=0}^{\infty}\rho_n(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$

The first term represents the direct path without interaction

$$\rho_0(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)=e^{i(1/\hbar)\int_{t_0}^{t}dt'L(\boldsymbol{x}',\boldsymbol{p}')}\delta^3(\boldsymbol{x}-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0))$$

and the contribution from a path containing N vertices will be

$$\rho_N(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)=\left(-\frac{i}{\hbar}\right)\int_{-\infty}^{c(t-t_1)>|\boldsymbol{x}-\boldsymbol{x}_1|}dt_1 e^{i(1/\hbar)\int_{t_0}^{t_1}dt'L(\boldsymbol{x}',\boldsymbol{p}')}\int d^3k_1V(\boldsymbol{k}_1)$$
$$\left\{\prod_{n=2}^{N-1}\left(-\frac{i}{\hbar}\right)\int_{t_{n-1}}^{c(t-t_n)>|\boldsymbol{x}-\boldsymbol{x}_n|}dt_n e^{i(1/\hbar)\int_{t_{n-1}}^{t_n}dt'L(\boldsymbol{x}',\boldsymbol{p}')}\int d^3k_nV(\boldsymbol{k}_n)\right\}$$
$$\left(-\frac{i}{\hbar}\right)\int_{t_{N-1}}^{c(t-t_N)>|\boldsymbol{x}-\boldsymbol{x}_N|}dt_N e^{i(1/\hbar)\int_{t_{N-1}}^{t_N}dt'L(\boldsymbol{x}',\boldsymbol{p}')}\int d^3k_NV(\boldsymbol{k}_N)e^{i(1/\hbar)\int_{t_N}^{t}dt'L(\boldsymbol{x}',\boldsymbol{p}')}\delta^3(\boldsymbol{x}-\boldsymbol{x}_N-\boldsymbol{v}_N(t-t_N))$$

By construction, the v_n for $n=0,1,\ldots,N$-1 are all less than c, but it may turn out that the resulting separation between $\boldsymbol{x}_N,t_N$ and $\boldsymbol{x},t$ would be spacelike. However, the delta function cancels such contributions because $v_N<c$ by construction, so we may make all the limits of the integrations over dt_n uniformly equal to t_{n-1} and t_{n+1} except that t_1 can come before t_0.

Note that ρ_N is a product of dimensionless terms and a delta function of position, so it has the dimensions of probability density. Since we have derived the Lagrangian and don’t make use of unphysical paths, we can hope the classical equations of motion will follow in appropriate limits.

We will argue below that conservation of energy also results from integrating over all possible interaction times t_n with $n>0$.

3. Conservation of Energy

We can conveniently parameterize the positions of the vertices in terms of vector parameters $\boldsymbol{a}_n$ where

$$\begin{aligned}\boldsymbol{v}_n &= \frac{\boldsymbol{a}_n}{t_{n+1}-t_n}+\boldsymbol{v}_0 \\ \boldsymbol{x}_{n+1} &= \boldsymbol{x}_n+\boldsymbol{v}_n\left(t_{n+1}-t_n\right) \\ &= \boldsymbol{x}_n+\boldsymbol{a}_n+\boldsymbol{v}_0\left(t_{n+1}-t_n\right)\end{aligned}$$

(letting $\boldsymbol{x}_{N+1}=\boldsymbol{x}$ and $t_{N+1}=t$) Thus $\boldsymbol{x}_0$ and $\boldsymbol{x}$ are reference points and quantized momentum is transferred only at the points $\boldsymbol{x}_1 \ldots \boldsymbol{x}_N$. These are related to the momentum that has been added at the *n*th vertex by

$$\begin{aligned}\boldsymbol{p}_n &= \frac{m\boldsymbol{v}_n}{\left\{1-v_n^2/c^2\right\}^{1/2}} \\ &= \frac{mc\left(\boldsymbol{a}_n+\frac{\boldsymbol{v}_0}{c}c\left(t_{n+1}-t_n\right)\right)}{\left\{\left(1-\frac{v_0^2}{c^2}\right)c^2\left(t_{n+1}-t_n\right)^2-2\left(\boldsymbol{a}_n\cdot\frac{\boldsymbol{v}_0}{c}\right)c\left(t_{n+1}-t_n\right)-\left|\boldsymbol{a}_n\right|^2\right\}^{1/2}}\end{aligned}$$

It is noteworthy that $\boldsymbol{p}_{n\perp}$ is just the component perpendicular to $\boldsymbol{v}_0$ of the sum of the momenta transferred up to and including the *n*th vertex.

$$\boldsymbol{p}_n = \hbar\boldsymbol{k}_n+\boldsymbol{p}_{n-1} = \sum_{m=1}^{n}\hbar\boldsymbol{k}_m+\boldsymbol{p}_0$$

$$\boldsymbol{p}_{n\perp} = \sum_{m=1}^{n}\left\{\hbar\boldsymbol{k}_m-\left(\hbar\boldsymbol{k}_m\cdot\hat{\boldsymbol{v}}_0\right)\hat{\boldsymbol{v}}_0\right\}$$

Therefore we can consider the following quadratic equation for $c\left(t_{n+1}-t_n\right)$

$$\left(1-\frac{v_0^2}{c^2}\right)c^2\left(t_{n+1}-t_n\right)^2-2\left(\boldsymbol{a}_n\cdot\frac{\boldsymbol{v}_0}{c}\right)c\left(t_{n+1}-t_n\right)-\left|\boldsymbol{a}_n\right|^2-m^2c^2\frac{a_{n\perp}^2}{\left(p_{n\perp}\right)^2}=0$$

with the solutions

$$\left(1-\frac{v_0^2}{c^2}\right)^{1/2} c\left(t_{n+1}-t_n\right)=\frac{\left(\boldsymbol{a}_n\cdot\frac{\boldsymbol{v}_0}{c}\right)}{\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}\pm\left\{\left|\boldsymbol{a}_n\right|^2+\frac{\left(\boldsymbol{a}_n\cdot\frac{\boldsymbol{v}_0}{c}\right)^2}{1-\frac{v_0^2}{c^2}}+\left(\frac{mca_{n\perp}}{p_{n\perp}}\right)^2\right\}^{1/2}$$

The plus sign is evidently needed for $t_{n+1}>t_n$, which must be true by construction except for *n*=0. Using this parameterization, we can now express the kinetic energy in terms of the component of momentum transferred perpendicular to $\boldsymbol{v}_0$.

$$\begin{aligned}\frac{T_n}{c}&=\frac{p_{n\perp}}{v_{n\perp}/c}\\&=\frac{p_{n\perp}}{a_{n\perp}}c\left(t_{n+1}-t_n\right)\\&=\frac{p_{n\perp}\left(\boldsymbol{a}_n\cdot\frac{\boldsymbol{v}_0}{c}\right)}{a_{n\perp}\left(1-\frac{v_0^2}{c^2}\right)}+\frac{mc}{\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}\left\{1+\left[\left|\boldsymbol{a}_n\right|^2+\frac{\left(\boldsymbol{a}_n\cdot\frac{\boldsymbol{v}_0}{c}\right)^2}{\left(1-\frac{v_0^2}{c^2}\right)}\right]\left(\frac{p_{n\perp}}{mca_{n\perp}}\right)^2\right\}^{1/2}\end{aligned}$$

The integral of the kinetic component of the Lagrangian takes a simple form when expressed in terms of the perpendicular momentum transfer.

$$\left(1-\frac{v_n^2}{c^2}\right)^{1/2} c\left(t_{n+1}-t_n\right)=mc\frac{a_{n\perp}}{p_{n\perp}}$$

Now we can also increment from all the segments of a path as follows

$$\begin{aligned}\hbar\varphi&=\int_{t_0}^{t}dt'L(\boldsymbol{x}',\boldsymbol{p}')\\&=L_0\left(t_1-t_0\right)+L_1\left(t_2-t_1\right)+\ldots+L_{N-1}\left(t_N-t_{N-1}\right)+L_N\left(t-t_N\right)\\&=L_0\left(t-t_0\right)+\left(L_1-L_0\right)\left(t_2-t_1\right)+\ldots\\&\quad+\left(L_{N-1}-L_0\right)\left(t_N-t_{N-1}\right)+\left(L_N-L_0\right)\left(t-t_N\right)\\&=L_0\left(t-t_0\right)+\hbar\sum_{n=1}^{N}\left(\overline{V}_n-\overline{V}_0\right)\left(t_{n+1}-t_n\right)+\sum_{n=1}^{N}\left(1-\frac{T_n}{T_0}\right)(mc)^2\frac{a_{n\perp}}{p_{n\perp}}\end{aligned}$$

The final result takes the form

$$\hbar\varphi = L_0\left(t-t_0\right) - \sum_{n=1}^{N}\frac{mc\left(\boldsymbol{a}_n\cdot\frac{\boldsymbol{v}_0}{c}\right)}{\left(1-\frac{v_0}{c^2}\right)^{1/2}} + \hbar\sum_{n=1}^{N}\left\{\left(\overline{V}_n-\overline{V}_0\right)\left(t_{n+1}-t_n\right)+\varphi_n\right\}$$

where we have used the following notation

$$\overline{V}_n\left(t_{n+1}-t_n\right) = \int_n^{n+1} dt' V\left(\boldsymbol{x}'\right)$$

$$\varphi_n = \frac{mc}{\hbar}\left(1-\left\{1+\left[1+\frac{\left(\boldsymbol{a}_n\cdot\hat{\boldsymbol{v}}_0/a_{n\perp}\right)^2}{\left(1-\frac{v_0^2}{c^2}\right)}\right]\left(\frac{p_{n\perp}}{mc}\right)^2\right\}^{1/2}\right)\frac{mca_{n\perp}}{p_{n\perp}}$$

Noting that $\boldsymbol{a}$ is the sum of the $\boldsymbol{a}_n$, which is the vector extending from the position that the particle would have reached at time *t* if it had continued on its original trajectory, to the actual end point $\boldsymbol{x}$.

$$\begin{aligned}L_0\left(t-t_0\right)-\sum_{n=1}^{N}\frac{mc\left(\boldsymbol{a}_n\cdot\frac{\boldsymbol{v}_0}{c}\right)}{\left(1-\frac{v_0^2}{c^2}\right)^{1/2}} &= \left\{mc^2\left(1-v_0^2/c^2\right)^{1/2}+\overline{V}_0\right\}\left(t-t_0\right)-m\frac{\left(\boldsymbol{x}-\boldsymbol{x}_0\right)\cdot\boldsymbol{v}_0-v_0^2\left(t-t_0\right)}{\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}\\ &= \left\{\frac{mc^2\left(1-v_0^2/c^2\right)}{\left(1-v_0^2/c^2\right)^{1/2}}+mc^2\frac{v_0^2/c^2}{\left(1-v_0^2/c^2\right)^{1/2}}+\overline{V}_0\right\}\left(t-t_0\right)-\boldsymbol{p}_0\cdot\left(\boldsymbol{x}-\boldsymbol{x}_0\right)\\ &= \left(T_0+\overline{V}_0\right)\left(t-t_0\right)-\boldsymbol{p}_0\cdot\left(\boldsymbol{x}-\boldsymbol{x}_0\right)\end{aligned}$$

so the phase becomes

$$\hbar\varphi = T_0\left(t-t_0\right)+\overline{V}_0\left(t-t_0\right)-\boldsymbol{p}_0\cdot\left(\boldsymbol{x}-\boldsymbol{x}_0\right)+\hbar\sum_{n=1}^{N}\left[\left(\overline{V}_n-\overline{V}_0\right)\left(t_{n+1}-t_n\right)+\varphi_n\right]$$

Thus assigning a consistent starting phase that cancels the initial energy term, $T_0+\overline{V}_0$ times t_0, and integrating over t_0, the coefficient of *t* resolves to $T_0+\overline{V}_0$ and otherwise the phase consists of integrals with upper limits cascading from -∞ to *t* and therefore having values that are independent of *t*.

On the other hand, by construction, the outgoing state is also a superposition of terms having phase $T_N+V\left(\boldsymbol{x}\right)$ times *t*. We conclude that the components with $T_N+V\left(\boldsymbol{x}\right)$ not equal to $T_0+\overline{V}_0$ must cancel and therefore total energy will be found to be conserved, but only to the degree that the initial value can be specified

4. Diffraction Example

We'll now consider application to the familiar double-slit diffraction problem in first order, where one quantum is exchanged between the particle and an aperture plate as shown in Figure 1. It will be assumed that the potential $V(\boldsymbol{x})$ is zero along all paths a particle might take.

Including the incoming phase term suggested by consistency, the first-order contribution to the phase from blocked paths will be

$$\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$

$$= -i(1/\hbar)\sum_s \int_{t_{1\min}(x_0,s)}^{t_{1\max}(x_0,s)} dt_1 e^{i(1/\hbar)L_0(t_1-t_0)} \iiint dk_{1x}dk_{1y}dk_{1z}V(\boldsymbol{k}_1)e^{i(1/\hbar)L_1(t-t_1)}\delta^3(\boldsymbol{x}-\boldsymbol{x}_1-\boldsymbol{v}_1(t-t_1))$$

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{v}_0\left(t_1 - t_0\right), \boldsymbol{p}_1 = \boldsymbol{p}_0 + \hbar\boldsymbol{k}_1, \boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{v}_1\left(t - t_1\right)$$

with

$$\int_{t_0}^{t_1} dt' L(\boldsymbol{x}',\boldsymbol{p}') = \int_{t_0}^{t_1} dt' \left\{ T(\boldsymbol{x}',\boldsymbol{p}') - p'\frac{dx'}{dt'} \right\} = T_0\left(t_1 - t_0\right) - \boldsymbol{p}_0 \cdot \left(\boldsymbol{x}_1 - \boldsymbol{x}_0\right)$$

$$\int_{t_1}^{t} dt' L(\boldsymbol{x}',\boldsymbol{p}') = \int_{t_1}^{t} dt' \left\{ T(\boldsymbol{x}',\boldsymbol{p}') - p'\frac{dx'}{dt'} \right\} = T_1\left(t - t_1\right) - \boldsymbol{p}_1 \cdot \left(\boldsymbol{x} - \boldsymbol{x}_1\right)$$

The vector $\boldsymbol{x}_0$ points to the position of the particle at time t_0, $\boldsymbol{v}_0$ is normal to the plane of the slits and is taken to be the +z-direction. Also, the *y*-direction is taken to be parallel to the slits. Since the aperture pattern is independent of+ *y*, and assuming the aperture is both extremely thin and impenetrable, we will assume the potential has the form

$$V\left(\boldsymbol{k}_1\right) = V\left(k_{1x}\right)\kappa\delta\left(k_{1y}\right)$$

$$\int d^3k_1 V\left(k_{1x}\right)\kappa\delta\left(k_{1y}\right)e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}} = V\left(x\right)\kappa\delta\left(z\right)$$

$$\int d^3k_1 V\left(k_1\right) = \int d^3k_1 V\left(k_{1x}\right)\kappa\delta\left(k_{1y}\right)$$

$$= \int dk_{1x}dk_{1z}V\left(k_{1x}\right)\kappa$$

Here κ is a constant, with the dimensions of a wave vector, that is unspecified in this approximation.

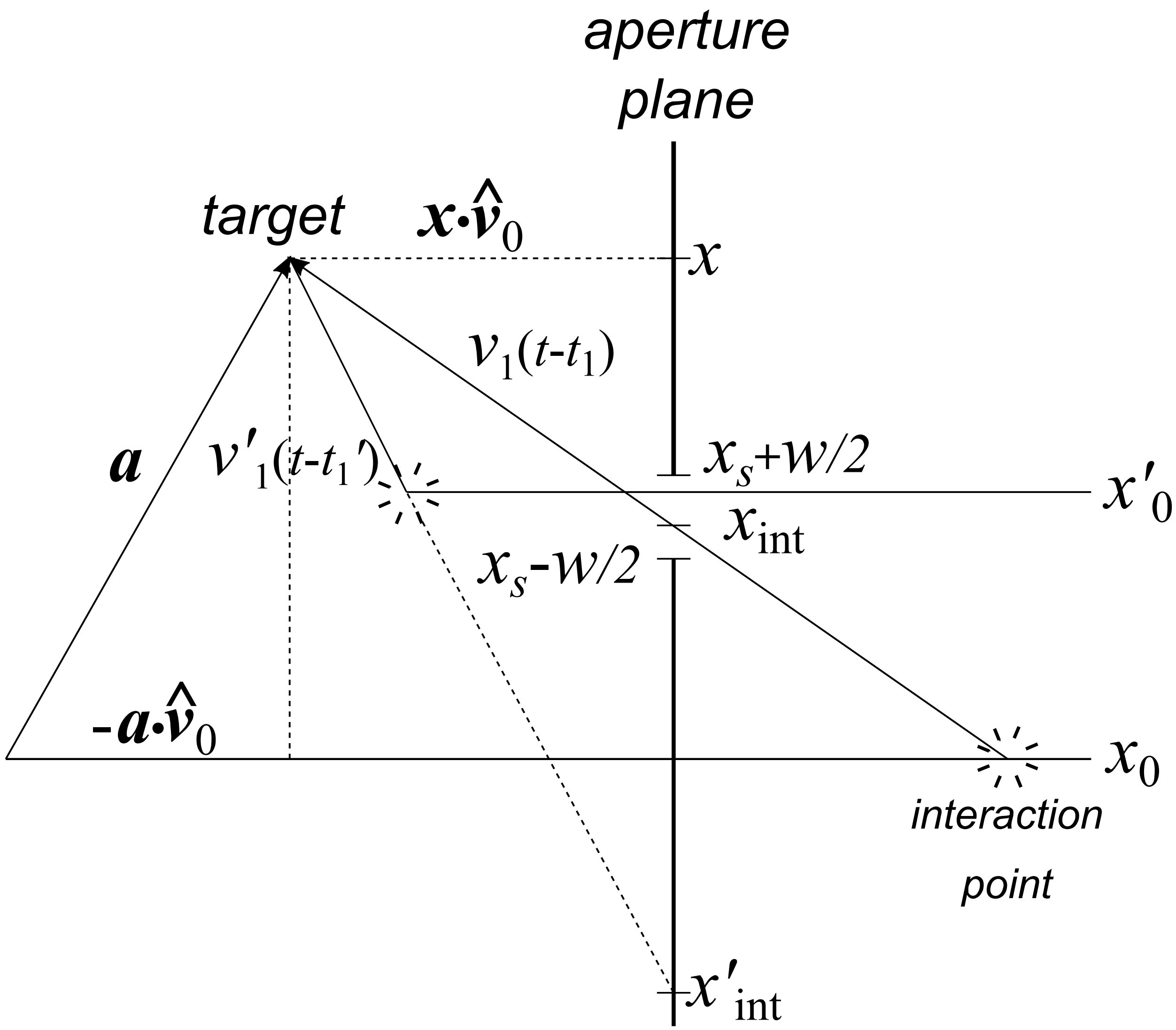

Figure 1. Possible paths of particles approaching normal to an *xy*-aperture plane, which contains a slit of width *w* parallel to the y-axis, and being redirected with one quantum exchange.

Thus integrating over dk_{1z} gives

$$\iiint dk_{1x}dk_{1y}dk_{1z}V(\boldsymbol{k}_1)e^{i(1/\hbar)L_1(t-t_1)}\delta^3(\boldsymbol{x}-\boldsymbol{x}_1-\boldsymbol{v}_1(t-t_1))$$
$$=\iint dk_{1x}dk_{1z}\kappa V(k_{1x})e^{i(1/\hbar)L_1(t-t_1)}\delta\left(x-x_0-v_{1x}(t-t_1)\right)\delta\left((\boldsymbol{x}-\boldsymbol{x}_1)\cdot\hat{\boldsymbol{v}}_0-v_{1z}(t-t_1)\right)\delta\left(y-y_1\right)$$
$$=\iint\left|\frac{\partial(k_{1x},k_{1z})}{\partial(v_{1x},v_{1z})}\right|dv_{1x}dv_{1z}\kappa V(k_{1x})e^{i(1/\hbar)L_1(t-t_1)}\delta\left(x-x_0-v_{1x}(t-t_1)\right)\delta\left((\boldsymbol{x}-\boldsymbol{x}_1)\cdot\hat{\boldsymbol{v}}_0-v_{1z}(t-t_1)\right)\delta\left(y-y_0\right)$$

because $\boldsymbol{k}_1$ must be in the plane perpendicular to the slits. Because of the delta functions, it is convenient to convert the double integration over $dk_{1x}dk_{1y}$ into equivalent integrations over $dv_{1x}dv_{1y}$. The Jacobian for this change of variables is found as follows

$$\hbar k_x = m\frac{v_{1x}}{\left(1-v_1^2/c^2\right)^{1/2}}$$

$$\hbar k_z = m\frac{v_{1z}}{\left(1-v_1^2/c^2\right)^{1/2}} - m\frac{v_0}{\left(1-v_0^2/c^2\right)^{1/2}}$$

$$\left|\frac{\partial(k_{1x},k_{1z})}{\partial(v_{1x},v_{1z})}\right| = \frac{m^2}{\hbar^2}\begin{vmatrix} \frac{1-v_{1z}^2/c^2}{(1-v_1^2/c^2)^{3/2}} & \frac{v_{1x}v_{1z}/c^2}{(1-v_1^2/c^2)^{3/2}} \\ \frac{v_{1x}v_{1z}/c^2}{(1-v_1^2/c^2)^{3/2}} & \frac{1-v_{1x}^2/c^2}{(1-v_1^2/c^2)^{3/2}} \end{vmatrix} = \frac{m^2}{\hbar^2}\frac{1-v_1^2/c^2}{(1-v_1^2/c^2)^3} = \frac{m^2}{\hbar^2}\frac{1}{(1-v_1^2/c^2)^2}$$

The delta functions take care of the integrations over dv_{1x} and dv_{1z} leaving

$$\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) = -i(1/\hbar)\sum_s \int_{t_{1\min}(x_0,t,s)}^{t_{1\max}(x_0,t,s)} dt_1 e^{i(1/\hbar)T_0(t_1-t_0)} \frac{(m/\hbar)^2\,\kappa V(k_{1x})}{(t-t_1)^2(1-v_1^2/c^2)^2} e^{i(1/\hbar)(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1} e^{i(1/\hbar)T_1(t-t_1)}\delta(y-y_0)$$

Unphysical paths for which $v_1 > c$ require unphysical values of $\hbar\boldsymbol{k}_1$ that cannot be present in the spectrum of the potential. By the way, since the input flux is assumed to be constant, the position $\boldsymbol{x}_0\cdot\hat{\boldsymbol{v}}_0$ of the particle on its trajectory at t_0 only defines a reference coordinate system and can be taken at the intersection of the original trajectory with the plane of the slits.

If the arriving flux of particles per unit area per unit time is *J*, then integrating over dx_0dy_0 and dt_0 gives the probability density of particles arriving at ***x*** at time *t* to be

$$\rho_{1B}(\boldsymbol{x},t;\boldsymbol{v}_0) = -i(1/\hbar)e^{i(1/\hbar)\{T_0(t-t_0)-\boldsymbol{p}_0\cdot(\boldsymbol{x}-\boldsymbol{x}_0)\}}\iint Jv_0dx_0dt_0 \int_{-\infty}^{c(t-t_1)\geq|x-\boldsymbol{x}_1|} dt_1 e^{i\varphi_1}\frac{(m/\hbar)^2\,\kappa V(k_{1x})}{(t-t_1)^2(1-v_1^2/c^2)^2}$$

where *κ* plays the role of a normalization constant (especially if we don't know the potential in full detail).

As above, we parameterize possible paths using a vector ***a*** that is in the plane of v_0 and the momentum $\hbar\boldsymbol{k}_1$ that is transferred such that

$$\boldsymbol{v}_1 = \boldsymbol{v}_0 + \frac{\boldsymbol{a}}{t - t_1}$$

Now consider integrating over t_0 or, equivalently, $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ first. The phase

$$\varphi_1 = \frac{mc}{\hbar}\left(1 - \left\{1 + \left[1 + \frac{\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 / a_x\right)^2}{\left(1 - \frac{v_0^2}{c^2}\right)}\right]\left(\frac{p_{1x}}{mc}\right)^2\right\}^{1/2}\right)\frac{mca_x}{p_{1x}}$$

$$= \frac{mca_x}{\hbar}\left(\frac{mc}{p_{1x}} - \left\{\left(\frac{mc}{p_{1x}}\right)^2 + \left[1 + \frac{\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 / a_x\right)^2}{\left(1 - \frac{v_0^2}{c^2}\right)}\right]\right\}^{1/2}\right)$$

will be stationary with respect to $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ when the derivative

$$\frac{d\varphi_1}{d\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)} = \frac{mca_x}{\hbar}\left(1 - \frac{1}{2}\left\{\left(\frac{mc}{p_{1x}}\right)^2 + \left[1 + \frac{\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 / a_x\right)^2}{\left(1 - \frac{v_0^2}{c^2}\right)}\right]\right\}^{-1/2} 2\left(\frac{mc}{p_{1x}}\right)\right)\frac{d}{d\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}\frac{mc}{p_{1x}}$$

$$-\frac{mca_x}{\hbar}\frac{1}{2}\left\{\left(\frac{mc}{p_{1x}}\right)^2 + \left[1 + \frac{\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 / a_x\right)^2}{\left(1 - \frac{v_0^2}{c^2}\right)}\right]\right\}^{-1/2} \frac{2\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 / a_x\right)}{\left(1 - \frac{v_0^2}{c^2}\right)a_x}$$

vanishes. The derivative can be evaluated with the help of

$$\frac{d}{d\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}\frac{mca_x}{p_{1x}} = \frac{1}{2}\left(\left(1 - \frac{v_0^2}{c^2}\right)c^2\left(t - t_1\right)^2 - 2\boldsymbol{a}\cdot\boldsymbol{v}_0\left(t - t_1\right) - a^2\right)^{-1/2}\left\{-2v_0\left(t - t_1\right) - 2\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right\}$$

$$= -\frac{p_{1x}}{mca_x}\left\{v_0\left(t - t_1\right) + \boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right\}$$

Thus a necessary condition that is met when the phase is stationary with respect to $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ becomes

$$\left\{1+\left[1+\frac{\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0/a_x\right)^2}{\left(1-\frac{v_0^2}{c^2}\right)}\right]\left(\frac{p_{1x}}{mc}\right)^2\right\}^{1/2}=\frac{v_0\left(t-t_1\right)+\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0-\frac{\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-\frac{v_0^2}{c^2}\right)}}{v_0\left(t-t_1\right)+\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}$$

Since the radical on the left-hand side of the last line must be (slightly) greater than unity, it turns out that $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ will have to be negative for the phase to be stationary.

Now instead of using the method of stationary phase and integrating directly over $d\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$, we will consider corresponding values of the final velocity v_1. This range turns out to be small enough to be a useful approximation to strict conservation of energy. At the same time, it will be convenient to suppress t_1 as a variable in favor of $x_{\mathbf{int}}$, which will be the position along the *x*-axis at which the path under consideration intersects the plane of the slits. Then relating similar triangles on Figure 1 gives

$$\begin{aligned}\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}&=\left\{\left(\frac{v_1\left(t-t_1\right)}{a_x}\right)^2-1\right\}^{1/2}\\&=\left\{1+\left(\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{a_x}\right)^2+2\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{a_x}\frac{v_0\left(t-t_1\right)}{a_x}+\left(\frac{v_0\left(t-t_1\right)}{a_x}\right)^2-1\right\}^{1/2}\\&=\left|\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{a_x}+\frac{v_0\left(t-t_1\right)}{a_x}\right|\end{aligned}$$

from which we are able to express $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0/a_x$ in terms of v_1 and $x_{\mathbf{int}}$ as follows

$$\begin{aligned}\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{a_x}&=-\frac{v_0\left(t-t_1\right)}{a_x}+\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}\\&=-\frac{v_0}{v_1}\frac{v_1\left(t-t_1\right)}{a_x}+\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}\\&=-\frac{v_0}{v_1}\left\{1+\left(\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}\right)^2\right\}^{1/2}+\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}\end{aligned}$$

using a result from similar triangles in Fig. 1. A different sign choice could have been made here, but it would have required the particle to pass through one of the slits to a position far behind the aperture plane before scattering. Therefore, $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ will be negative in the remaining cases of interest provided

$$\frac{v_0}{v_1}\frac{\left\{\left(x-x_{\text{int}}\right)^2+\left(\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right)^2\right\}^{1/2}}{\left(x-x_{\text{int}}\right)}\geq\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}$$

$$\frac{v_1}{v_0}\leq\left\{1+\left(\frac{x-x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^2\right\}^{1/2}$$

Thus the maximum velocity will be only very slightly greater than v_0 in a typical experiment designed to show the fringes in the distribution.

There is also a minimum in v_1 as a function of $t-t_1$ that can be determined as follows

$$\begin{aligned}\frac{dv_1^2}{d\left(t-t_1\right)}&=-2\frac{a^2}{\left(t-t_1\right)^3}-2\frac{\boldsymbol{a}\cdot\boldsymbol{v}_0}{\left(t-t_1\right)^2}\\&=-\frac{2}{\left(t-t_1\right)^2}\left\{\frac{a^2}{\left(t-t_1\right)}+\boldsymbol{a}\cdot\boldsymbol{v}_0\right\}=0\end{aligned}$$

If $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ is positive, $v_1^2-v_0^2$ starts out large and positive at small $t-t_1$ and decreases monotonically to 0 when $t-t_1$ is large. If $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ is negative, however, v_1^2 goes through a minimum at $t-t_1=-a^2/\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ and returns to v_0^2 at large $t-t_1$. This minimum value of v_1 is equal to

$$\begin{aligned}t-t_{1\min}&=-\frac{a^2}{\boldsymbol{a}\cdot\boldsymbol{v}_0}\\v_{1\min}^2&=a^2\left(-\frac{\boldsymbol{a}\cdot\boldsymbol{v}_0}{a^2}\right)^2+2\boldsymbol{a}\cdot\boldsymbol{v}_0\left(-\frac{\boldsymbol{a}\cdot\boldsymbol{v}_0}{a^2}\right)+v_0^2\\&=\frac{v_0^2}{1+\left(\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{a_x}\right)^2}\end{aligned}$$

Then using the relation of $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0/a_x$ to $\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0/\left(x-x_{\text{int}}\right)$ and v_1/v_0, we find

$$v_{1\min}^2\left(1+\left(\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}-\frac{v_0}{v_{1\min}}\left\{1+\left(\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}\right)^2\right\}^{1/2}\right)^2\right)=v_0^2$$

$$\left(\left\{1+\left(\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}\right)^2\right\}^{1/2}\frac{v_{1\min}}{v_0}-\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}\right)^2=0$$

Because this lower limit of v_1 is also very close to v_0, it will be useful to change the integration variable from t_0 or $\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0$ to v_1. The resulting range in v_1 becomes

$$\left\{1+\left(\frac{x-x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^2\right\}^{-1/2}\le\frac{v_1}{v_0}\le\left\{1+\left(\frac{x-x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^2\right\}^{1/2}$$

and it is noted that the difference between the limits

$$\frac{v_{1\max}-v_{1\min}}{v_0}=\frac{\left(\dfrac{x-x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^2}{\left\{1+\left(\dfrac{x-x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^2\right\}^{1/2}}$$

is second order in $\left(x-x_{\text{int}}\right)/\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0$, which can already be very small in some cases of interest.

Now suppose the aperture plane is in the *z*=0 plane and the slits are parallel to the *y*-axis. With one change in direction, the particle can reach a blocked position $\boldsymbol{x}$ from a range of positions on initial trajectories having $x_0 < x$ as shown in Figure 1.

The range of the integration over x_0 depends only on the position of the top edge of the slit in question, assuming that the final trajectory extends to x_{int} when the initial trajectory passes through the slit above x_{int}. The limits of the integral over x_{int} depend only on the slit in question and the limits of the integration over T_1 or v_1 depend only on x_{int}.

To change the integration variables t_1 and t_0 to v_1 and x_{int}, we need to calculate another Jacobian using the definitions

$$x - x_{\text{int}} = \frac{a_x \boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0}{v_0 (t - t_1) + \boldsymbol{a} \cdot \hat{\boldsymbol{v}}_0}$$

$$v_1^2 = \left(\frac{\boldsymbol{a}}{t - t_1} + \boldsymbol{v}_0 \right)^2$$

The partial derivatives result in

$$\frac{\partial (v_1, x_{\text{int}})}{\partial (t_1, t_0)} = \frac{1}{v_1} \begin{vmatrix} \boldsymbol{v}_1 \cdot \dfrac{\boldsymbol{a}}{(t - t_1)^2} & \boldsymbol{v}_1 \cdot \dfrac{\boldsymbol{v}_0}{(t - t_1)} \\ \dfrac{a_x \boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0}{\left[v_0 (t - t_1) + \boldsymbol{a} \cdot \hat{\boldsymbol{v}}_0 \right]^2} (v_0) & -\dfrac{a_x \boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0}{\left[v_0 (t - t_1) + \boldsymbol{a} \cdot \hat{\boldsymbol{v}}_0 \right]^2} (v_0) \end{vmatrix}$$

$$= -\frac{a_x \boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0}{\left[a_x \boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0 / (x - x_{\text{int}}) \right]^2} \frac{v_1^2}{v_1 (t - t_1)}$$

Using similar triangles in Figure 1 to show

$$\frac{v_1 (t - t_1)}{a_x} = \frac{\left\{ (x - x_{\text{int}})^2 + (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0)^2 \right\}^{1/2}}{x - x_{\text{int}}}$$

we find the Jacobian further simplifies to

$$\frac{\partial (v_1, x_{\text{int}})}{\partial (t_1, t_0)} = -\frac{a_x \boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0 v_0}{\left[a_x \boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0 / (x - x_{\text{int}}) \right]^2} \frac{v_1^2 (x - x_{\text{int}})}{a_x \left\{ (x - x_{\text{int}})^2 + (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0)^2 \right\}^{1/2}}$$

$$= -v_1^2 \left(\frac{x - x_{\text{int}}}{a_x} \right) \left(\frac{x - x_{\text{int}}}{\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} \right) \frac{v_0}{a_x \left\{ 1 + \left(\dfrac{\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0}{x - x_{\text{int}}} \right)^2 \right\}^{1/2}}$$

With the simultaneous change of variables, the complex probability density becomes

$$\rho_{1B}(\boldsymbol{x}, t; \boldsymbol{v}_0) = i(1/\hbar) e^{i(1/\hbar)\left\{ T_0 (t - t_0) - \boldsymbol{p}_0 \cdot (\boldsymbol{x} - \boldsymbol{x}_0) \right\}} \sum_s J$$

$$\times \iint dx_{\text{int}} dx_0 \int_{v_{1\min}}^{v_{1\max}} dv_1 \frac{(\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0)(m/\hbar)^2}{v_0 (1 - v_1^2 / c^2)(x - x_{\text{int}}) \left\{ (x - x_{\text{int}})^2 + (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0)^2 \right\}^{1/2}} e^{i\varphi_1} \kappa V(k_{1x})$$

We will now express the phase φ_1 in terms of v_1 and the intercept $\mathbf{x_{int}}$ starting with the *x*-component of the momentum

$$\frac{p_{1x}\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)}{mca_x\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}=\frac{\frac{v_1}{c}\frac{v_0}{c}}{\left(1-v_1^2/c^2\right)^{1/2}\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}\left\{\left\{1+\left(\frac{x-x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^2\right\}^{-1/2}-\frac{v_0}{v_1}\right\}$$

which is (can be) negative provided v_1 is less than v_0 as determined above. Therefore the complicated radical becomes

$$\left\{1+\left[1+\frac{\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0/a_x\right)^2}{\left(1-\frac{v_0^2}{c^2}\right)}\right]\left(\frac{p_{1x}}{mc}\right)^2\right\}^{1/2}=\frac{\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}{\left(1-\frac{v_1^2}{c^2}\right)^{1/2}}-\frac{p_{1x}\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)}{mca_x\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}$$

$$=\frac{1}{\left(1-v_1^2/c^2\right)^{1/2}\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}\left\{1-\frac{\frac{v_1}{c}\frac{v_0}{c}}{\left\{1+\left(\frac{x-x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^2\right\}^{1/2}}\right\}$$

This is also positive. And so finally the extra phase, written in terms of the intercept and the final velocity, simplifies to

$$\varphi_1=\left\{\frac{\left(1-\frac{v_0^2}{c^2}\right)^{1/2}\left(1-\frac{v_1^2}{c^2}\right)^{1/2}-1}{\frac{v_0}{c}\frac{v_1}{c}}\left\{1+\left(\frac{x-x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^2\right\}^{1/2}+1\right\}\frac{p_0a_x}{\hbar}\frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x-x_{\text{int}}}$$

As a function of v_1, φ_1 passes through an extremum when the derivative

$$\frac{d}{d\left(\frac{v_1}{c}\right)}\left\{\frac{\left(1-\frac{v_0^2}{c^2}\right)^{1/2}\left(1-\frac{v_1^2}{c^2}\right)^{1/2}-1}{\frac{v_0}{c}\frac{v_1}{c}}\right\}=\frac{c^3}{v_0v_1^2}\left\{\frac{\left(1-\frac{v_1^2}{c^2}\right)^{1/2}-\left(1-\frac{v_0^2}{c^2}\right)^{1/2}}{\left(1-\frac{v_1^2}{c^2}\right)^{1/2}}\right\}$$

vanishes, which happens when energy is exactly conserved. The slope is positive when v_1 is less than v_0 so φ_1 is going through a maximum at $v_1=v_0$. However the value of φ_1 is negative at that point if x_{int} is not equal to *x*. If they were equal, it would mean that the target position is uncovered, which is not being considered at this time.

To first order order in $v_1/c - v_0/c$, the phase becomes

$$\hbar\varphi_1 \approx \left\{ -\left\{ 1 + \left(\frac{x - x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0} \right)^2 \right\}^{1/2} + 1 \right\} p_0 a_x \frac{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{x - x_{\text{int}}}$$

Corrections start at second order in $v_1 - v_0$. However, the allowed range of $v_1 - v_0$ is itself second order in $(x - x_{\text{int}})/\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0$, so the variation of the phase becomes fourth order in $(x - x_{\text{int}})/\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0$ and will be negligible in some cases of interest. In such cases, integration over dv_1 then just replaces v_1 by v_0 and multiplies the integrand by $\left((x - x_{\text{int}})/\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right)^2 v_0^2$ giving

$$\rho_{1B}(\boldsymbol{x},t;\boldsymbol{v}_0) \approx i(1/\hbar) e^{i(1/\hbar)\{T_0(t-t_0) - \boldsymbol{p}_0\cdot(\boldsymbol{x}-\boldsymbol{x}_0)\}} \sum_s J \iint dx_{\text{int}} dx_0 \left(\frac{x - x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0} \right)^2 v_0^2$$
$$\times \frac{(m/\hbar)^2}{v_0\left(1 - v_0^2/c^2\right)(x - x_{\text{int}}) \left\{ 1 + \left(\frac{x - x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0} \right)^2 \right\}^{1/2}} \kappa V(k_{1x}) \exp\left\{ -\frac{i}{2\hbar} \frac{x - x_{\text{int}}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0} p_0 a_x \right\}$$

Now the Fourier transform of the potential will be assumed to be adequately approximated by

$$\begin{aligned} V(k_{1x}) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx' V(x') e^{-ik_{1x}x'} \\ &= V_0\delta(k_{1x}) - \frac{V_0}{2\pi} \int_{slits} dx' e^{-ik_{1x}x'} \\ &= V_0\delta(k_{1x}) + \frac{V_0}{k_{1x}\pi} \left\{ e^{ik_{1x}L/2} \sin k_{1x}(W/2) + e^{-ik_{1x}L/2} \sin k_{1x}(W/2) \right\} \\ &= V_0\delta(k_{1x}) + \frac{2V_0}{k_{1x}\pi} \cos(k_{1x}L/2) \sin(k_{1x}W/2) \end{aligned}$$

and we are assuming $k_{1x} = 0$ is not included. That is, the target position is not exposed to the direct rays, so the delta function can be ignored. Using the definition of k_{1x} with $v_1 = v_0$

$$\hbar k_{1x} = \frac{m v_0}{\left(1 - v_0^2 / c^2\right)^{1/2}} \frac{x - x_{\text{int}}}{\left\{\left(x - x_{\text{int}}\right)^2 + \left(\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0\right)^2\right\}^{1/2}}$$

the complex probability density simplifies to

$$\rho_{1B}(\boldsymbol{x}, t; \boldsymbol{v}_0) \approx i(1/\hbar) e^{i(1/\hbar)\left\{T_0(t - t_0) - \boldsymbol{p}_0 \cdot (\boldsymbol{x} - \boldsymbol{x}_0)\right\}} \sum_s J \frac{(m/\hbar) 2\kappa V_0}{\left(1 - v_0^2 / c^2\right)^{1/2} \pi\left(\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0\right)}$$
$$\times \iint dx_{\text{int}} dx_0 \cos\left(k_{1x} L / 2\right) \sin\left(k_{1x} W / 2\right) \exp\left\{-\frac{i}{2\hbar} \frac{x - x_{\text{int}}}{\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} p_0 a_x\right\}$$

If the interaction occurs in front of the plane of the apertures, x_{int} varies over the slits and x_0 varies from $-\infty$ up to x_{int}. Integrating over dx_0 in the infinite range is not well defined. However, one could integrate over dx_{int} first over the range in which x_0 is below the bottom of the slit being considered. The remaining double integral with both $x_0 < x_{\text{int}}$ in the slit is well defined.

On the other hand, if the interaction occurs after the plane of the slits, x_{int} will vary from $-\infty$ up to x_0 while x_0 will vary over the slit. Here we split the integral over dx_{int} into a first part from $-\infty$ up to the bottom of the slit and a second part within the slit. The first part is well defined by integrating over dx_0 first. The remaining double integral with $x_{\text{int}} < x_0$ is well defined and combines with the double integral above with $x_0 < x_{\text{int}}$. Schematically we then have, per slit *s*

$$\iint dx_{\text{int}} dx_0 \cos\left(k_{1x} L/2\right) \sin\left(k_{1x} W/2\right) \exp\left\{-\frac{i}{\hbar} \frac{x - x_{\text{int}}}{2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} p_0 a_x\right\}$$

$$= \left\{ \int_{x_s - w/2}^{x_s + w/2} dx_{\text{int}} \int_{-\infty}^{x_{\text{int}}} dx_0 + \int_{x_s - w/2}^{x_s + w/2} dx_0 \int_{-\infty}^{x_0} dx_{\text{int}} \right\} \cos\left(k_{1x} L/2\right) \sin\left(k_{1x} W/2\right) \exp\left\{-\frac{i}{\hbar} \frac{x - x_{\text{int}}}{2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} p_0 a_x\right\}$$

$$= \left\{ \begin{array}{l} \left\{ \int_{x_s - w/2}^{x_s + w/2} dx_{\text{int}} \int_{-\infty}^{x_s - w/2} dx_0 + \int_{x_s - w/2}^{x_s + w/2} dx_{\text{int}} \int_{x_s - w/2}^{x_{\text{int}}} dx_0 \right\} \\ + \left\{ \int_{x_s - w/2}^{x_s + w/2} dx_0 \int_{-\infty}^{x_s - w/2} dx_{\text{int}} + \int_{x_s - w/2}^{x_s + w/2} dx_0 \int_{x_s - w/2}^{x_0} dx_{\text{int}} \right\} \end{array} \right\} \cos\left(k_{1x} L/2\right) \sin\left(k_{1x} W/2\right) \exp\left\{-\frac{i}{\hbar} \frac{x - x_{\text{int}}}{2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} p_0 a_x\right\}$$

$$= \left\{ \begin{array}{l} \int_{x_s - w/2}^{x_s + w/2} dx_{\text{int}} \int_{-\infty}^{x_s - w/2} dx_0 + \int_{x_s - w/2}^{x_s + w/2} dx_{\text{int}} \int_{x_s - w/2}^{x_s + w/2} dx_0 \\ + \int_{x_s - w/2}^{x_s + w/2} dx_0 \int_{-\infty}^{x_s - w/2} dx_{\text{int}} \end{array} \right\} \cos\left(k_{1x} L/2\right) \sin\left(k_{1x} W/2\right) \exp\left\{-\frac{i}{\hbar} \frac{x - x_{\text{int}}}{2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} p_0 a_x\right\}$$

$$= \left\{ \int_{-\infty}^{x_s - w/2} dx_0 \int_{x_s - w/2}^{x_s + w/2} dx_{\text{int}} + \int_{-\infty}^{x_s + w/2} dx_{\text{int}} \int_{x_s - w/2}^{x_s + w/2} dx_0 \right\} \cos\left(k_{1x} L/2\right) \sin\left(k_{1x} W/2\right) \exp\left\{-\frac{i}{\hbar} \frac{x - x_{\text{int}}}{2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} p_0 a_x\right\}$$

where the integrations with finite limits that don't depend on the other variable can be done first so that the integrations over the infinite limits will be defined consistently. Performing the integrals with finite limits first gives

$$\iint dx_{\text{int}} dx_0 \cos\left(k_{1x} L/2\right) \sin\left(k_{1x} W/2\right) \exp\left\{-\frac{i}{\hbar} \frac{x - x_{\text{int}}}{2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} p_0 a_x\right\}$$

$$= \left\{ \int_{-\infty}^{x_s - w/2} dx_0 \int_{x_s - w/2}^{x_s + w/2} dx_{\text{int}} + \int_{-\infty}^{x_s + w/2} dx_{\text{int}} \int_{x_s - w/2}^{x_s + w/2} dx_0 \right\} \sum_{\varepsilon_L = \pm 1} \frac{1}{2} \sum_{\varepsilon_W = \pm 1} \frac{\varepsilon_W}{i2} \exp\left\{ i k_0 \left(x - x_{\text{int}}\right) \left(\varepsilon_L L + \varepsilon_W W - \left(x - x_0\right)\right) / 2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0 \right\}$$

$$= \sum_{\varepsilon_L = \pm 1} \frac{1}{2} \sum_{\varepsilon_W = \pm 1} \frac{\varepsilon_W}{i2} \left\{ \begin{array}{l} \int_{-\infty}^{x_s - w/2} dx_0 \left. \dfrac{\exp\left\{ -i k_0 \left(x - x_{\text{int}}\right) \left(x - x_0 - \varepsilon_L L - \varepsilon_W W\right) / 2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0 \right\}}{i k_0 \left(x - x_0 - \varepsilon_L L - \varepsilon_W W\right) / 2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} \right|_{x_{\text{int}} = x_s - w/2}^{x_s + w/2} \\ + \int_{-\infty}^{x_s + w/2} dx_{\text{int}} \left. \dfrac{\exp\left\{ -i k_0 \left(x - x_{\text{int}}\right) \left(x - x_0 - \varepsilon_L L - \varepsilon_W W\right) / 2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0 \right\}}{i k_0 \left(x - x_{\text{int}}\right) / 2\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0} \right|_{x_0 = x_s - w/2}^{x_s + w/2} \end{array} \right\}$$

Here we have approximated k_{1x} by $k_0\left(x - x_{\textbf{int}}\right) / \boldsymbol{x} \cdot \hat{\boldsymbol{v}}_0$ using the tangent of the scattering angle instead of the sine. This error will be small in some cases of interest and is probably due to approximations made in integrating over dv_1. The first pair of integrals, taken together, are well behaved even when the denominator vanishes, because

$$\left.\frac{\exp\left\{-ik_0\left(x-x_{\text{int}}\right)\left(x-x_0-\varepsilon_L L-\varepsilon_W W\right)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right\}}{ik_0\left(x-x_0-\varepsilon_L L-\varepsilon_W W\right)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right|_{x_{\text{int}}=x_s-w/2}^{x_s+w/2}$$

$$=2\frac{\sin k_0\frac{w}{2}\left(x-x_0-\varepsilon_L L-\varepsilon_W W\right)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{k_0\left(x-x_0-\varepsilon_L L-\varepsilon_W W\right)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\exp\left\{-ik_0\left(x-x_s\right)\left(x-x_0-\varepsilon_L L-\varepsilon_W W\right)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right\}$$

Otherwise, with changes of the integration variables, and assuming they remain positive over the entire range of the integrations, we have

$$\iint dx_{\text{int}}dx_0\cos\left(k_{1x}L/2\right)\sin\left(k_{1x}W/2\right)\exp\left\{-\frac{i}{\hbar}\frac{x-x_{\text{int}}}{2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}p_0a_x\right\}$$

$$=\frac{2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}{k_0}\sum_{\varepsilon_L=\pm1}\frac{1}{2}\sum_{\varepsilon_W=\pm1}\frac{\varepsilon_W}{i2}\left\{\begin{array}{l}-\int\limits_{k_0\left(x-\left(x_s-w/2\right)\right)\left(x-\left(x_s-w/2\right)-\varepsilon_L L-\varepsilon_W W\right)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}^{\infty}du\frac{\exp\{-iu\}}{iu}\\+\int\limits_{k_0\left(x-\left(x_s+w/2\right)\right)\left(x-\left(x_s+w/2\right)-\varepsilon_L L-\varepsilon_W W\right)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}^{\infty}du\frac{\exp\{-iu\}}{iu}\end{array}\right\}$$

where a cancelation has taken place between the upper limit of the first integral and the lower limit of the second.

Now it can be noted that the integrals being considered are all of the form

$$\begin{aligned}\int\limits_z^\infty d\xi\,\frac{e^{-i\xi}}{\xi}&=\int\limits_z^\infty d\xi\,\frac{\cos\xi}{\xi}=-i\int\limits_z^\infty d\xi\,\frac{\sin\xi}{\xi}\\&=-Ci(z)+i\left\{Si(z)-\frac{\pi}{2}\right\}\\&=-f(z)\sin(z)+g(z)\cos(z)+i\left\{-f(z)\cos(z)-g(z)\sin(z)\right\}\\&=\left\{g(z)-if(z)\right\}e^{-iz}\\&\approx-i\frac{e^{-iz}}{z}\end{aligned}$$

where Ci and Si are the Cosine Integral and Sine Integral functions as defined by Wohlfram [4] and we use asymptotic forms for them given by NIST [5] in lowest order.

$$
\begin{aligned}
S_i(z) &= \frac{\pi}{2} - f(z)\cos(z) - g(z)\sin(z) \\
C_i(z) &= f(z)\sin(z) - g(z)\cos(z) \\
f(z) &\approx \frac{1}{z}\left(1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \frac{6!}{z^6} + \ldots\right) \\
g(z) &\approx \frac{1}{z^2}\left(1 - \frac{3!}{z^2} + \frac{5!}{z^4} - \frac{7!}{z^6} + \ldots\right)
\end{aligned}
$$

We will approximate the series by the first term since z takes the form

$$
\begin{aligned}
z &= \frac{p_0\left(x - x_s - \beta\right)}{\hbar 2\,\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\left(x - x_s - \alpha\right) \\
&= \frac{p_0}{\hbar 2\,\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\left\{x^2 - x\left(2x_s + \alpha + \beta\right) + \left(x_s + \alpha\right)\left(x_s + \beta\right)\right\} \\
&= k_0\,\frac{x^2}{2\,\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0} - k_0\,\frac{x}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\left(x_s + \frac{\alpha+\beta}{2}\right) + k_0\,\frac{\left(x_s+\alpha\right)\left(x_s+\beta\right)}{2\,\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}
\end{aligned}
$$

We assume that x is chosen to be large compared to the scale of the slits in order to make the diffraction pattern easier to observe. The leading term in the phase of order x^2 will then be relatively large but the same for the various combinations of x_s, α and β. The second term comprises the average x-component of the final momentum times various combinations of the slit parameters. These combinations will be compared to the expected diffraction pattern, which is proportional to the square of the Fourier transform of the potential. The quadratic terms in the slit-parameters will be relatively small because we choose the slit parameters to make x relatively large at the first minimum of the diffraction pattern.

Noting that w=W, the parameters for the two integrals are

$$
\begin{aligned}
\alpha &= -W/2 \\
\beta &= -W/2 + \varepsilon_L L + \varepsilon_W W \\
sign &= -\varepsilon_W
\end{aligned}
$$

for the upper set of integrals and

$$
\begin{aligned}
\alpha &= W/2 \\
\beta &= W/2 + \varepsilon_L L + \varepsilon_W W \\
sign &= \varepsilon_W
\end{aligned}
$$

for the lower group. Including the sum over slits at $x_s = \pm L/2$, there are a total of 16 terms.

Sign	x_s	ε_L	ε_W	$x_s+(\alpha+\beta)/2$
-1	*L/2*	1	1	*L*
1	*L/2*	1	-1	*L-W*
-1	*L/2*	-1	1	0
1	*L/2*	-1	-1	*-W*
-1	*-L/2*	1	1	0
1	*-L/2*	1	-1	*-W*
-1	*-L/2*	-1	1	*-L*
1	*-L/2*	-1	-1	*-L-W*
1	*L/2*	1	1	*L+W*
-1	*L/2*	1	-1	*L*
1	*L/2*	-1	1	*W*
-1	*L/2*	-1	-1	0
1	*-L/2*	1	1	*W*
-1	*-L/2*	1	-1	0
1	*-L/2*	-1	1	*-L+W*
-1	*-L/2*	-1	-1	*-L*

Table I. Spatial frequencies (in units of $k_0/2x\cdot\hat{v}_0$) of surviving terms after integration of complex probability density over dx_0 and dx_{int}.

The signs and weights of the spatial frequencies shown in Table I are reproduced by the formula

$$\sum_{x_s=\pm L/2}\sum_{\varepsilon_L=\pm1}\sum_{\varepsilon_W=\pm1}\varepsilon_W\left\{\begin{array}{l}-\displaystyle\int_{k_0(x-(x_s-w/2))(x-(x_s-w/2)-\varepsilon_L L-\varepsilon_W W)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}^{\infty}du\,\frac{\exp\{-iu\}}{iu}\\+\displaystyle\int_{k_0(x-(x_s+w/2))(x-(x_s+w/2)-\varepsilon_L L-\varepsilon_W W)/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}^{\infty}du\,\frac{\exp\{-iu\}}{iu}\end{array}\right\}$$

$$\approx -i\left(k_0\frac{x^2}{2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)^{-1}e^{-ik_0x^2/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\left(e^{ik_0xL/\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}+2+e^{-ik_0xL/\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)\left(e^{ik_0xW/\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}-2+e^{-ik_0xW/\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)$$

Thus we find that the complex probability density simplifies to

$$\rho_{1B}(\boldsymbol{x},t;\boldsymbol{v}_0) \approx (1/\hbar)e^{i(1/\hbar)\{T_0(t-t_0)-\boldsymbol{p}_0\cdot(\boldsymbol{x}-\boldsymbol{x}_0)\}} \frac{2(m/\hbar)\kappa V_0 J}{\left(1-v_0^2/c^2\right)^{1/2}\pi} e^{-ik_0\frac{x^2}{2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}}$$
$$\times\left\{\frac{\left(e^{ik_0 xL/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}+e^{-ik_0 xL/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)\left(e^{ik_0 xW/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}-e^{-ik_0 xW/2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)}{\left(\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right)^{1/2}\left(k_0\frac{x}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\right)}\right\}^2$$

which, except for its overall phase, has the same form as given by standard quantum theory under similar conditions. Relating the experimental distribution to the absolute magnitude of the complex probability density seems to be better motivated than Born's rule is in the standard theory.

5. Conclusion

By expanding the domain of probability to complex numbers, quantum interference can be seen to arise from the classical understanding of force as the time-rate of change of momentum and the un-classical quantization of momentum transfers in units of $\hbar$ times the wave vector $\boldsymbol{k}$ of the Fourier transform of the potential energy. A complex probability density is thereby calculated and it is strongly suggested that particle detectors respond to its absolute magnitude. No mysterious wave function needing further interpretation is required.

Further development of a quantum-mechanical theory based on complex probability should be pursued. In that event, Youssef's [6] application of complex probabilities to a Bayesian foundation for quantum mechanics appears to be promising. As Feynman [7] once said, "… *Nature with her infinite imagination has found another set of principles for determining probabilities…*"

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