Towards a Foundational Principle for Quantum Mechanics

The wind is not moving, the flag is not moving.
Mind is moving.[1]

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A simple foundational principle for quantum mechanics is proposed leading naturally to a complex probability density expressed as a sum of path integrals. The action is then derived in its relativistically-invariant form with the Lagrangian following from the principle instead of being imposed as an extra hypothesis. However, the sum is restricted to physical paths on which the particle has real vector momentum. This restriction evidently conflicts with the usual derivation of Schroedinger's equation and from it conservation of energy and probability as defined by Born. To see if the restricted path integral sum leads to sensible predictions, the double-slit experiment is analyzed using a single-quantum transfer approximation. The result is found to be in general agreement with experiment provided we add a natural measurement hypothesis. However, the energy of the particle is not necessarily conserved.. The maximum change in energy is a loss that is proportional to the square of Planck's constant when expressed in terms of the slit geometry and the mass of the particle. This bounding energy change is largest for light particles at low energy and might be observable with electrons at around 10 eV.

Introduction

When I first heard about Heisenberg's uncertainty principle, I naively supposed that the statistical nature of the predictions of quantum mechanics would emerge from the unpredictability of individual exchanges of quanta between fields and material particles. However, the development didn't take this path. What ensued was best characterized by Richard Feynman when he said [1] "we cannot make the mystery go away … we will just tell you how it works."

After thinking about it for a long time, I may have found a simple way forward along the path I had originally anticipated. The first part of the foundational principal (notably missing from conventional quantum theory [2]) that I want to propose is:

a) **Particles and fields interact discretely through quantized exchanges of momentum that occur at predictable rates but at unpredictable times.**

I should add that I think this is obvious because one could only discover the timing of a first exchange with additional interactions that would compound the situation.

1. Building on the Foundation

The force acting on a particle is the negative of the gradient of the potential energy. When the potential, say $V(\boldsymbol{x})$, is expressed as a Fourier integral, the integrand differs from that of the potential by a factor of $-i\boldsymbol{k}$,

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[1] *Zen Flesh, Zen Bones - A Collection of Zen and pre-Zen Writing*s compiled by Paul Reps, Doubleday, New York, 1961, p. 114.

$$\begin{aligned} \boldsymbol{F} &= -\nabla V(\boldsymbol{x}) \\ &= -i(1/\hbar)\int d^3k\left(\hbar \boldsymbol{k}\right)V(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \end{aligned}$$

On the other hand, $\hbar\boldsymbol{k}$ is the momentum per quanta exchanged with the field. What is left, formally at least, should be the rate of exchange of such quanta in d^3k, and integrating this rate over $\boldsymbol{k}$-space therefore suggests that the total rate of exchange at all wave vectors is $-i(1/\hbar)V(\boldsymbol{x})$. Formally applying Poisson statistics and exponentiating the negative of the expected number of exchanges in a time dt then gives a "probability"

$$P_a = e^{i(1/\hbar)V(\boldsymbol{x})dt}$$

that no exchange between a particle at $\boldsymbol{x}$ and the field occurs in time dt. (The subscript just indicates that we are not done defining the total probability yet.)

Now, we still need a term that represents the effects of inertia. If the potential energy is a measure of the average rate at which quanta are exchanged with the field, the mechanical energy could be similarly related to interactions with whatever it is that gives particles their mass and inertia. Adding the total mechanical energy T to the potential V gives:

$$P_b = e^{i(1/\hbar)(V(\boldsymbol{x})+T(\boldsymbol{p}))dt}$$

for the probability of no exchange in time dt. Here $\boldsymbol{p}$ is the momentum of the particle and $T(\boldsymbol{p})$ is the total (kinetic + rest) mechanical energy. Now consider a particle starting from location $\boldsymbol{x}_0$ at time t_0 with momentum $\boldsymbol{p}_0$. The probability density for finding it still on its initial trajectory at some later time t_1 will be:

$$\rho_c(\boldsymbol{x}_1,t_1;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) = e^{i(1/\hbar)\int_0^1 dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}_0))}\delta^3\left(x_1-\boldsymbol{x}_0-\boldsymbol{v}_0\left(t_1-t_0\right)\right)$$

because multiplying the probabilities for each increment of time results in adding the differential terms in the exponent. They add up to the path integral of the sum of the potential and mechanical energies. Of course the initial momentum and position can't be known simultaneously because instruments that might be used to determine them are also unpredictable at the quantum level. Nevertheless, the classical metaphysical principle that a particle has a unique position and momentum is not necessarily abandoned. Owing to our unavoidable ignorance, we will just have to average over position or momentum or some combination thereof in a practical situation.

Continuing with the formal application of Poisson statistics suggests that the differential probability of exchange of a quantum with wave vector $\boldsymbol{k}_1$ in d^3k_1 during a time dt_1 is

$$d^4P_d = -i(1/\hbar)dt_1 d^3k_1 V(\boldsymbol{k}_1)e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1+i(1/\hbar)(V(\boldsymbol{x}_1)+T(\boldsymbol{p}_1))dt_1}$$

After picking up momentum $\hbar\boldsymbol{k}_1$, the particle will be on a new trajectory until the next quantum is exchanged. The amplitude for arriving at some final position $\boldsymbol{x}$ at time t can be expressed as an integral along the initial trajectory of a product of three terms. The first term is the probability for not interacting along the initial leg, which is given by P_c if we say the first quantum is exchanged at t_1. The second term is the amplitude for the first exchange, which modifies the trajectory the particle will follow to the second exchange:

$$\begin{aligned}\boldsymbol{p}_1 &= \boldsymbol{p}_0 + \hbar\boldsymbol{k}_1,\\ \boldsymbol{x}_2 &= \boldsymbol{x}_1 + (t_2 - t_1)\boldsymbol{v}_1\end{aligned}$$

The third term is the amplitude for arriving at the final position $\boldsymbol{x}$ at time t with starting position shifted to $\boldsymbol{x}_1$ and t_1. This reasoning suggests an integral equation

$$\begin{aligned}&\rho(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)\\ &= e^{i(1/\hbar)\int_{t_0}^{t} dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}_0))}\delta^3(x-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0)) - i(1/\hbar)\int_{t_0}^{t} dt_1 e^{i(1/\hbar)\int_0^1 dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}_0))}\int d^3k_1 V(\boldsymbol{k}_1)e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1}\rho(\boldsymbol{x},\boldsymbol{t};\boldsymbol{x}_1,\boldsymbol{t}_1,\boldsymbol{v}_1)\\ &\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{v}_0(t-t_1)\end{aligned}$$

where it should be noted that $d^3kV(\boldsymbol{k})$ has the dimensions of energy, and the effective integration over the space-time coordinates is restricted by relativistic causality. In other words, the intervals between $\boldsymbol{x},t$ and $\boldsymbol{x}_1,t_1$ and between $\boldsymbol{x}_1,t_1$ and $\boldsymbol{x}_0,t_0$ must be time-like.

We can rewrite the phase factor that comes from the Fourier transform accurately as:

$$\begin{aligned}i\boldsymbol{k}_1\cdot\boldsymbol{x}_1 &= (i/\hbar)(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1\\ &= (i/\hbar)\int_0^1 d\boldsymbol{p}'\cdot\boldsymbol{x}'\end{aligned}$$

because the increment in momentum is concentrated at the end of the interval where $\boldsymbol{x}'=\boldsymbol{x}_1$. This term combines with the path integral of the energy to produce a factor of the form:

$$A = e^{i(1/\hbar)\int_0^1 dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}'))+d\boldsymbol{p}'\cdot\boldsymbol{x}'}$$

which appears to be closely related to the Feynman path integral [3]. In fact, if we integrate the third term in the integrand by parts once, we find

$$\begin{aligned}A &= e^{i(1/\hbar)(\boldsymbol{p}_1\cdot\boldsymbol{x}_1-\boldsymbol{p}_0\cdot\boldsymbol{x}_0)}e^{i(1/\hbar)\int_0^1 dt\left(V(\boldsymbol{x}')+T(\boldsymbol{p}')-\boldsymbol{p}'\cdot\boldsymbol{v}'\right)}\\ &= e^{i(1/\hbar)(\boldsymbol{p}_1\cdot\boldsymbol{x}_1-\boldsymbol{p}_0\cdot\boldsymbol{x}_0)}e^{i(1/\hbar)\int_0^1 dt' L(\boldsymbol{x}',\boldsymbol{p}')}\end{aligned}$$

contains Feynman's path integral of the Lagrangian $L(\boldsymbol{x}, \boldsymbol{p})$ multiplied by

$$e^{i(1/\hbar)(\boldsymbol{p}_1\cdot\boldsymbol{x}_1-\boldsymbol{p}_0\cdot\boldsymbol{x}_0)}$$

Now, the velocity between t_0 and t_1 is $\boldsymbol{v}_0$, and $\hbar\boldsymbol{k}_1=\boldsymbol{p}_1-\boldsymbol{p}_0$, so we can rearrange terms to express the equation in a potentially simpler form:

$$e^{i(1/\hbar)\boldsymbol{p}_0\cdot\boldsymbol{x}_0}\rho(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)=$$

$$e^{i(1/\hbar)\int_{t_0}^{t}dt'(V(\boldsymbol{x}')+T(\boldsymbol{p}_0))+i(1/\hbar)\boldsymbol{p}_0\cdot\boldsymbol{x}_0}\delta^3(\boldsymbol{x}-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0))-i(1/\hbar)\int_{t_0}^{t}dt_1 e^{i(1/\hbar)\int_0^1 dt' L(\boldsymbol{x}',\boldsymbol{p}')}\int d^3k_1 V(\boldsymbol{k}_1)e^{i(1/\hbar)\boldsymbol{p}_1\cdot\boldsymbol{x}_1}\rho(\boldsymbol{x},\boldsymbol{t};\boldsymbol{x}_1,\boldsymbol{t}_1,\boldsymbol{v}_1)$$

$$\boldsymbol{x}_1=\boldsymbol{x}_0+\boldsymbol{v}_0\left(t_1-t_0\right),\ \boldsymbol{p}_1=\boldsymbol{p}_0+\hbar\boldsymbol{k}_1,\ \boldsymbol{x}=\boldsymbol{x}_1+\boldsymbol{v}_1\left(t-t_1\right)$$

This can be viewed as an integral equation for the probability density for the particle to arrive at $\boldsymbol{x}$ and t multiplied by a phase factor associated with the (hypothetical) initial momentum and position. We are left with the path integral of the Lagrangian and the Fourier transform of the potential to deal with in actual calculations.

As in Feynman’s formulation, contributions from nearby paths should add in phase when the Lagrangian is close to an extremum and otherwise cancel. However, only physical paths are generated by considering quantized transfers of momentum. Feynman's formulation includes all paths in coordinate space along which time does not regress. This means that the particle is required to move faster than the speed of light on some of these paths. On the other hand, the expression for the Lagrangian that results from our hypotheses

$$L=V(\boldsymbol{x})+T(\boldsymbol{p})-\boldsymbol{p}\cdot\boldsymbol{v}$$

while correct classically, because $\boldsymbol{p}\cdot\boldsymbol{v}$ is equal to twice the kinetic energy, also gives the correct equations of motion when the energy and momentum are corrected according to the special theory of relativity. In fact, the integral of the Lagrangian appears in its Lorentz invariant form

$$\left(T(\boldsymbol{p})+V(\boldsymbol{x})-\boldsymbol{p}\cdot\boldsymbol{v}\right)dt=p_\mu dx^\mu$$

It may be that the inclusion of unphysical paths in Feynman's formulation, seeing that it is then compatible with Schroedinger's equation, is just a very convenient approximation. For example in solving the double-slit experiment, it is easier to use solutions of Schroedinger's equation than to compute the path integral explicitly. However, matching up solutions of Schroedinger's equation in different regions forces the energy with which the particle reaches the screen to be exactly the same as it had approaching the barrier. It seems that exact conservation of energy ultimately follows from the inclusion of unphysical paths in the path-integral formulation, and calculating the amplitude as a sum restricted to physical paths could give a slightly different answer.

At least one more hypothesis is necessary in order to be able to predict the results of measurements. Since our complex probability density is related to Feynman's path integral, minus the unphysical paths, the simplest choice is to proceed as in standard quantum mechanics. That, is the measurement probability density should be proportional to the square of our complex probability density. However, the integral of this density over the spatial coordinates might not be constant in time if Schroedinger's equation is not satisfied. Therefore it seems to be necessary to consider transient rather than steady-state events and integrate over time to get the measurement density:

> **b.) The probability density of a measurement is proportional to the integral over time of the square of the absolute magnitude of the complex probability density for the system being measured to arrive at the point of the measurement.**

This procedure seems to be reasonable provided we consider events that are isolated in time or sufficiently infrequent so that interference effects between successive particles in the recording media become negligible. This, in turn, may impose an experimental upper limit on the time-scale for isolated events in practical situations.

Since we have already derived the Lagrangian, we can say that the classical equations of motion follow from our quantum hypothesis even if it departs in some respects from the predictions of conventional quantum mechanics. That there is some departure seems clear from the fact we do not include unphysical paths and do not expect Schroedinger's equation to hold exactly.

To clarify whether or not our quantum principle can agree with experiment, it seems appropriate next to investigate the familiar double-slit experiment.

2. Quantum Mystery – Diffraction by Slits

Feynman used the double-slit experiment as an example and discussed it in great detail because, he said, "it contains the only mystery." The "only" mystery, of course, is that particles appear to behave like waves in this experiment. This wave-like behavior appears as interference fringes in the distribution of particles that pass through the slits vs. the angle between the incoming and outgoing momentum of the particles. Feynman used the double-slit experiment as a pedagogical tool. Since the time of his lectures, the expected interference behavior has been confirmed directly in elegant experiments [4].

Exact solutions to the equation developed above are not known yet. When the path leads to a collision with the screen, the expansion may converge slowly, if at all, but the result must be small if $\boldsymbol{x}$ is on the other side of the screen from $\boldsymbol{x}_0$. When there is no collision, we may be able to consider only the lowest-order terms. The complex probability density that a particle on a trajectory passing through $\boldsymbol{x}_0$, t_0 will arrive at $\boldsymbol{x},t$ in this approximation is

$$
\begin{aligned}
&\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)\\
&= -i(1/\hbar)\sum_s \int_{t_{1\min}(x_0,s)}^{t_{1\max}(x_0,s)} dt_1 e^{i(1/\hbar)T_0(t_1-t_0)} \iiint dk_{1x}dk_{1y}dk_{1z}V(\boldsymbol{k}_1)e^{i(1/\hbar)(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1} e^{i(1/\hbar)T_1(t-t_1)}\delta^3(\boldsymbol{x}-\boldsymbol{x}_1-\boldsymbol{v}_1(t-t_1))\\
&\boldsymbol{x}_1 = \boldsymbol{x}_0+\boldsymbol{v}_0\left(t_1-t_0\right), \boldsymbol{p}_1 = \boldsymbol{p}_0+\hbar\boldsymbol{k}_1, \boldsymbol{x} = \boldsymbol{x}_1+\boldsymbol{v}_1\left(t_1-t_0\right)
\end{aligned}
$$

provided $\boldsymbol{x}$ is not on the original path. For each slit s, the target position $\boldsymbol{x}$ is visible from the position of the particle between the limits in the integration over t_1.

The integration over t_1 includes times when the particle can scatter *before* passing through one of the slits on its way to the observation point $\boldsymbol{x}$. That is, the lower limit $t_{1\min}$ is generally less than t_0. The upper limit occurs when the target is no longer visible from the position of the particle. The window does not close, however, if the initial trajectory passes through one of the slits. In that case the upper limit is determined by relativistic causality instead.

For simplicity, let's assume the particle initially approaches the plane of the slits along the normal and call this direction the z-axis. The potential is actually independent of position along the axis parallel to the slits, which we'll call the y-axis. The dependence of the Fourier transform of the potential on the corresponding component of the wave vector will therefore be concentrated into a delta function at k_{1y}=0. Thus the outgoing momentum will remain in the original plane perpendicular to the direction of the slits. Under these assumptions, the first-order term above simplifies to

$$
\begin{aligned}
&\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)\\
&=-i(1/h)\sum_s \int_{t_{1\min}(x_0,s)}^{t_{1\max}(x_0,s)} dt_1 e^{i(1/\hbar)T_0(t_1-t_0)} \iint dk_{1x}dk_{1z}V(k_{1x},k_{1z})e^{i(1/\hbar)(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1} e^{i(1/\hbar)T_1(t-t_1)}\delta^3(\boldsymbol{x}-\boldsymbol{x}_1-\boldsymbol{v}_1(t-t_1))\\
&\boldsymbol{p}_1=\boldsymbol{p}_0+\hbar\boldsymbol{k}_1\\
&k_{1y}=0
\end{aligned}
$$

Because of the delta functions, it is convenient to convert the double integration over $dk_{1x}dk_{1y}$ into equivalent integrations over $dv_{1x}dv_{1z}$. The Jacobian for this change of variables is

$$
\begin{aligned}
\left|\frac{\partial(k_{1x},k_{1z})}{\partial(v_{1x},v_{1z})}\right| &= \begin{vmatrix} \dfrac{\partial k_{1x}}{\partial v_{1x}} & \dfrac{\partial k_{1z}}{\partial v_{1x}} \\[2ex] \dfrac{\partial k_{1x}}{\partial v_{1z}} & \dfrac{\partial k_{1z}}{\partial v_{1z}} \end{vmatrix}\\
&= \frac{m}{\hbar^2}\begin{vmatrix} \dfrac{1-v_{1z}^2/c^2}{(1-v_1^2/c^2)^{3/2}} & \dfrac{v_{1x}v_{1z}/c^2}{(1-v_1^2/c^2)^{3/2}} \\[2ex] \dfrac{v_{1x}v_{1z}/c^2}{(1-v_1^2/c^2)^{3/2}} & \dfrac{1-v_{1x}^2/c^2}{(1-v_1^2/c^2)^{3/2}} \end{vmatrix}\\
&= \frac{m}{\hbar^2(1-v_1^2/c^2)^2}
\end{aligned}
$$

The delta functions take care of the integrations leaving

$$
\begin{aligned}
&\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) \\
&= -i(m/2\pi\hbar^3)\sum_s \int_{t_{1\min}(x_0,t,s)}^{t_{1\max}(x_0,t,s)} dt_1 e^{i(1/\hbar)T_0(t_1-t_0)} \frac{V(k_{1x},k_{1z})}{(t-t_1)^2(1-v_1^2/c^2)^2} e^{i(1/\hbar)(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1} e^{i(1/\hbar)T_1(t-t_1)}\delta(y-y_0) \\
&\boldsymbol{p}_1 = \boldsymbol{p}_0 + \hbar\boldsymbol{k}_1 \\
&\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{v}_1(t-t_1)
\end{aligned}
$$

where the upper limit of the integration over t_1 can be set by relativistic causality. This will be the case if the initial trajectory passes through one of the openings.

Let's also assume the plane from which the slits has been cut is very thin and the z-dependence of the potential contains a delta function at z_1=0. Then the Fourier transform of the potential is actually independent of k_{1z} so the probability density can be written

$$
\begin{aligned}
&\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) \\
&= -i(m/2\pi\hbar^3)\sum_s \int_{t_{1\min}(x_0,t,s)}^{t_{1\max}(x_0,t,s)} dt_1 e^{i(1/\hbar)T_0(t_1-t_0)} \frac{V(k_{1x})}{(t-t_1)^2(1-v_1^2/c^2)^2} e^{i(1/\hbar)(\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot\boldsymbol{x}_1} e^{i(1/\hbar)T_1(t-t_1)}\delta(y-y_0) \\
&\boldsymbol{p}_1 = \boldsymbol{p}_0 + \hbar\boldsymbol{k}_1 \\
&\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{v}_1(t-t_1)
\end{aligned}
$$

2.1 Equation for Stationary Phase

Next, we anticipate that contributions to the integral will be small except when the phase of the integrand is stationary. The easiest way to see when that happens is to note that

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{v}_0(t_1-t_0)$$

This allows the phase in the integrand (in units of $\hbar$) to be expressed in the particularly simple form

$$
\begin{aligned}
\hbar\phi &= T_0(t_1-t_0) + (\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot(\boldsymbol{x}_0+\boldsymbol{v}_0(t_1-t_0)) + T_1(t-t_1) \\
&= (L_0+\boldsymbol{p}_0\cdot\boldsymbol{v}_0)(t_1-t_0) + (\boldsymbol{p}_1-\boldsymbol{p}_0)\cdot(\boldsymbol{x}_0+\boldsymbol{v}_0(t_1-t_0)) + (L_1+\boldsymbol{p}_1\cdot\boldsymbol{v}_1)(t-t_1) \\
&= L_0(t-t_0) - \boldsymbol{p}_0\cdot\boldsymbol{x}_0 + \boldsymbol{p}_1\cdot\boldsymbol{x} + (L_1-L_0)(t-t_1)
\end{aligned}
$$

This phase is stationary with respect to the time of the interaction when

$$
\begin{aligned}
\hbar\frac{d\phi}{dt_1} &= \frac{d\boldsymbol{p}_1}{dt_1}\cdot\boldsymbol{x} + (L_0-L_1) + \frac{dL_1}{dt_1}(t-t_1) \\
&= 0
\end{aligned}
$$

The derivative of the Lagrangian is given by

$$\frac{dL_1}{dt_1} = -\frac{T_1}{c^2}\boldsymbol{v}_1 \cdot \frac{d\boldsymbol{v}_1}{dt_1}$$

and the derivative of the momentum is related to the velocity and its rate of change by

$$\begin{aligned}\frac{d\boldsymbol{p}_1}{dt_1} &= \frac{T_1}{c^2}\frac{d\boldsymbol{v}_1}{dt_1} + m\boldsymbol{v}_1\frac{d}{dt_1}(1-v_1^2/c^2)^{-1/2} \\ &= \frac{T_1}{c^2}\frac{d\boldsymbol{v}_1}{dt_1} + \frac{T_1}{c^4(1-v_1^2/c^2)}\left(\boldsymbol{v}_1 \cdot \frac{d\boldsymbol{v}_1}{dt_1}\right)\boldsymbol{v}_1\end{aligned}$$

The condition for stationary phase can therefore be written

$$\left(\frac{T_1}{c^2}\frac{d\boldsymbol{v}_1}{dt_1} + \frac{T_1}{c^4(1-v_1^2/c^2)}\left(\boldsymbol{v}_1 \cdot \frac{d\boldsymbol{v}_1}{dt_1}\right)\boldsymbol{v}_1\right)\cdot \boldsymbol{x} + (L_0 - L_1) - \frac{T_1}{c^2}\boldsymbol{v}_1 \cdot \frac{d\boldsymbol{v}_1}{dt_1}(t-t_1) = 0$$

$$\frac{T_1}{c^2}\frac{d\boldsymbol{v}_1}{dt_1}\cdot\left(\boldsymbol{x} - \boldsymbol{v}_1(t-t_1) + \frac{1}{(1-v_1^2/c^2)}\left(\frac{\boldsymbol{v}_1}{c}\cdot\boldsymbol{x}\right)\frac{\boldsymbol{v}_1}{c}\right) + (L_0 - L_1) = 0$$

The velocity $\boldsymbol{v}_1$ after the interaction can be conveniently written as

$$\begin{aligned}v_{1x} &= \frac{x - x_0}{(t-t_1)} \\ v_{1z} &= \frac{z - z_0 - v_0(t_1 - t_0)}{(t-t_1)} \\ \boldsymbol{v}_1 &= \frac{\boldsymbol{x} - (\boldsymbol{x}_0 + \boldsymbol{v}_0(t-t_0)) + \boldsymbol{v}_0(t-t_1)}{(t-t_1)} \\ &= \frac{\boldsymbol{x} - (\boldsymbol{x}_0 + \boldsymbol{v}_0(t-t_0))}{(t-t_1)} + \boldsymbol{v}_0 \\ &= \frac{\boldsymbol{a}}{t-t_1} + \boldsymbol{v}_0\end{aligned}$$

where we have introduced the vector $\boldsymbol{a}$ pointing to the final position $\boldsymbol{x}$ from the position $\boldsymbol{x}_0+\boldsymbol{v}_0(t-t_0)$, which is where the particle would be at time t if it continued on its original course without interacting. (See Figure 1. for the relations between these vectors.) In turn, the rate of change of the components of $\boldsymbol{v}_1$ are straightforwardly calculated to be

$$\frac{d\boldsymbol{v}_1}{dt_1} = \frac{\boldsymbol{x} - \boldsymbol{x}_0 - (t - t_0)\boldsymbol{v}_0}{(t - t_1)^2}$$
$$= \frac{\boldsymbol{a}}{(t - t_1)^2}$$

Thus the rate of change of $\boldsymbol{v}_1$ with respect to t_1 is constant in direction $\boldsymbol{a}$**,** but it diverges in magnitude as t_1 approaches t. We can use this to rewrite the condition for the phase to be stationary as

$$\frac{\boldsymbol{a}}{(c(t - t_1))^2} \cdot \left(\boldsymbol{x} - \boldsymbol{v}_1(t - t_1) + \frac{1}{(1 - v_1^2 / c^2)} \left(\frac{\boldsymbol{v}_1}{c} \cdot \boldsymbol{x} \right) \frac{\boldsymbol{v}_1}{c} \right) + \frac{(L_0 - L_1)}{T_1} = 0$$

If the interaction occurs very late, $\boldsymbol{v}_1$ can become appreciable compared to the speed of light c. Otherwise, $c(t\text{-}t_1)$ will be large compared to other quantities in the equation with the dimensions of length. This means that the first term of the equation will be small, and the solution will be found near $L_0=L_1$. There will be a solution (for fixed t) because the second term starts out negative (if we include times t_1 before t_0), passes through zero when energy is conserved exactly, and then becomes positive for later values of t_1.

The condition for stationary phase can be rewritten as

$$\left(1 - \frac{v_0^2}{c^2}\right)^{1/2} \left(1 - \frac{v_1^2}{c^2}\right)^{1/2} = 1 - \frac{v_1^2}{c^2} - \frac{\boldsymbol{a}}{(c(t - t_1))^2} \cdot \left(\boldsymbol{x} - \boldsymbol{v}_1(t - t_1) + \frac{1}{(1 - v_1^2 / c^2)} \left(\frac{\boldsymbol{v}_1}{c} \cdot \boldsymbol{x} \right) \frac{\boldsymbol{v}_1}{c} \right)$$

Squaring this, and simplifying somewhat, leads to

$$\left(\frac{v_1^2}{c^2} - \frac{v_0^2}{c^2} \right) = -2 \frac{\mathbf{a}}{c(t - t_1)} \cdot \frac{\mathbf{x}_1 + \boldsymbol{\varepsilon}}{c(t - t_1)} + \frac{1}{(1 - v_1^2 / c^2)} \left(\frac{\mathbf{a}}{c(t - t_1)} \cdot \frac{\mathbf{x}_1 + \boldsymbol{\varepsilon}}{c(t - t_1)} \right)^2$$

$$\frac{\mathbf{a}}{c(t - t_1)} \cdot \left(\frac{\mathbf{v}_1}{c} + \frac{\mathbf{v}_0}{c} \right) = -2 \frac{\mathbf{a}}{c(t - t_1)} \cdot \frac{\mathbf{x}_1 + \boldsymbol{\varepsilon}}{c(t - t_1)} + \frac{1}{(1 - v_1^2 / c^2)} \left(\frac{\mathbf{a}}{c(t - t_1)} \cdot \frac{(\mathbf{x}_1 + \boldsymbol{\varepsilon})}{c(t - t_1)} \right)^2$$

$$\frac{\mathbf{a}}{c(t - t_1)} \cdot \left(\frac{\mathbf{v}_1}{c} + \frac{\mathbf{v}_0}{c} + 2 \frac{\mathbf{x}_1 + \boldsymbol{\varepsilon}}{c(t - t_1)} - \frac{1}{(1 - v_1^2 / c^2)} \frac{(\mathbf{x}_1 + \boldsymbol{\varepsilon})}{c(t - t_1)} \left[\frac{\mathbf{a}}{c(t - t_1)} \cdot \frac{(\mathbf{x}_1 + \boldsymbol{\varepsilon})}{c(t - t_1)} \right] \right) = 0$$

$$\frac{\mathbf{a}}{c(t - t_1)} \cdot \left(2 \frac{\mathbf{b} + \boldsymbol{\varepsilon}}{c(t - t_1)} - \frac{1}{(1 - v_1^2 / c^2)} \frac{\mathbf{x}_1 + \boldsymbol{\varepsilon}}{c(t - t_1)} \left[\frac{\mathbf{a}}{c(t - t_1)} \cdot \frac{\mathbf{x}_1 + \boldsymbol{\varepsilon}}{c(t - t_1)} \right] \right) = 0$$

where we introduce the definitions

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{1}{(1-v_1^2/c^2)}\left(\frac{\boldsymbol{v}_1}{c}\cdot\boldsymbol{x}\right)\frac{\boldsymbol{v}_1}{c} \\
&= \frac{1}{(1-v_1^2/c^2)}\left(\frac{\boldsymbol{v}_1}{c}\times\left(\frac{\boldsymbol{v}_1}{c}\times\boldsymbol{x}\right)+\frac{v_1^2}{c^2}\boldsymbol{x}\right) \\
\boldsymbol{b} &= \frac{1}{2}\left(\boldsymbol{v}_1+\boldsymbol{v}_0\right)(t-t_1)+\boldsymbol{x}_1 \\
&= \boldsymbol{x}_1+\boldsymbol{v}_0(t-t_1)+\frac{\boldsymbol{a}}{2} \\
&= \boldsymbol{x}_0+\boldsymbol{v}_0(t-t_0)+\frac{\boldsymbol{a}}{2}
\end{aligned}
$$

for the vectors $\boldsymbol{b}$, which is another vector that is constant in direction, and $\boldsymbol{\varepsilon}$, which is small compared to $\boldsymbol{b}$ in the non-relativistic limit. Provided t-t_1 is not infinite, the stationary-phase condition becomes equivalent to

$$
2\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon}) = \frac{1}{1-v_1^2/c^2}\left(\frac{\boldsymbol{a}\cdot(\boldsymbol{x}_1+\boldsymbol{\varepsilon})}{c(t\text{-}t_1)}\right)^2
$$

The right-hand side is always positive, but it will remain very small as long as $c(t$-$t_1)$ is large. Consequently $\boldsymbol{a}$ must be nearly orthogonal to $\boldsymbol{b}+\boldsymbol{\varepsilon}$. Even if $|\boldsymbol{a}|$ becomes very small, $\boldsymbol{a}$ will still have to be nearly orthogonal to $\boldsymbol{b}+\boldsymbol{\varepsilon}$ because the right-hand side scales like $|\boldsymbol{a}|^2$. If, to form the non-relativistic limit, we neglect both $\boldsymbol{a}\cdot\boldsymbol{\varepsilon}$ and the right-hand side altogether, we will have

$$
\begin{aligned}
2\boldsymbol{a}\cdot\boldsymbol{b} &= \left(\boldsymbol{x}-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0)\right)\cdot\left(\boldsymbol{x}_0+\boldsymbol{v}_0(t-t_0)+\frac{1}{2}\boldsymbol{a}\right) \\
&= \left(\boldsymbol{x}-\boldsymbol{x}_0-\boldsymbol{v}_0(t-t_0)\right)\cdot\left(\boldsymbol{x}+\boldsymbol{x}_0+\boldsymbol{v}_0(t-t_0)\right) \\
&= |\boldsymbol{x}|^2-\left|\boldsymbol{x}_0+\boldsymbol{v}_0(t-t_0)\right|^2 = 0
\end{aligned}
$$

At this level of approximation, the phase is stationary at a particular value of t and is independent of t_1. This suggests that the path integral of the Lagrangian may not always vary as rapidly from its stationary value as is usually assumed.

2.2 Solution Assuming Energy is Conserved Exactly

To get a better approximation, it will be necessary to handle certain small quantities delicately. Anticipating that energy will be at least approximately conserved, we first note that

$$\frac{v_1^2}{c^2} = \left(\frac{\boldsymbol{a}}{c(t-t_1)} + \frac{\boldsymbol{v}_0}{c}\right)^2$$
$$= \frac{\boldsymbol{a}\cdot\left(\boldsymbol{a}+2\boldsymbol{v}_0(t-t_1)\right)}{\left(c(t-t_1)\right)^2} + \frac{v_0^2}{c^2}$$
$$= \frac{2\boldsymbol{a}\cdot\left(\boldsymbol{b}-\boldsymbol{x}_1\right)}{\left(c(t-t_1)\right)^2} + \frac{v_0^2}{c^2}$$

This shows that square of the velocity, and hence the energy, will be changed very little by the interaction, provided $c(t\text{-}t_1)$ is large compared to any of the quantities in the problem with the dimension of length. We can also see that energy would be conserved exactly if $\boldsymbol{a}$ were orthogonal to the vector $\boldsymbol{b}\text{-}\boldsymbol{x}_1$, which gives a value of $t_1\text{-}t_0$

$$t_1 - t_0 = \frac{\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{x}_0)}{\boldsymbol{a}\cdot\boldsymbol{v}_0}$$

This is equivalent to

$$t - t_1 = t - t_0 - (t_1 - t_0)$$
$$= t - t_0 - \frac{\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{x}_0)}{\boldsymbol{a}\cdot\boldsymbol{v}_0}$$
$$= -\frac{a^2}{2\boldsymbol{a}\cdot\boldsymbol{v}_0}$$

which provides one relation between the remaining two free parameters t and t_1.

Now, if $\boldsymbol{a}\cdot\boldsymbol{b}$ is equal to $\boldsymbol{a}\cdot\boldsymbol{x}_1$, then it follows that $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})$ is equal to $\boldsymbol{a}\cdot(\boldsymbol{x}_1+\boldsymbol{\varepsilon})$**.** Using this in the equation for stationary phase gives

$$c(t-t_1) = \left(\frac{\boldsymbol{a}\cdot\left(\boldsymbol{x}_1+\boldsymbol{\varepsilon}\right)}{2(1-v_1^2/c^2)}\right)^{1/2}$$

if it is permissible to divide by $\boldsymbol{a}\cdot(\boldsymbol{x}_1+\boldsymbol{\varepsilon})$. However, this result is evidently incompatible with non-relativistic values of v_1 (which is equal to v_0 in the special case under consideration).

In the non-relativistic limit, $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})$ and $\boldsymbol{a}\cdot(\boldsymbol{x}_1+\boldsymbol{\varepsilon})$ must both be zero if the equation for stationary phase is satisfied and energy is exactly conserved. This condition specifies one of the two remaining unknowns, x_0 and $t\text{-}t_0$ (or $\boldsymbol{a}\cdot\boldsymbol{v}_0/v_0$). After multiplying through by $(1\text{-}v_1^2/c^2)[c(t\text{-}t_1)]^2$, the equation for stationary phase simplifies to

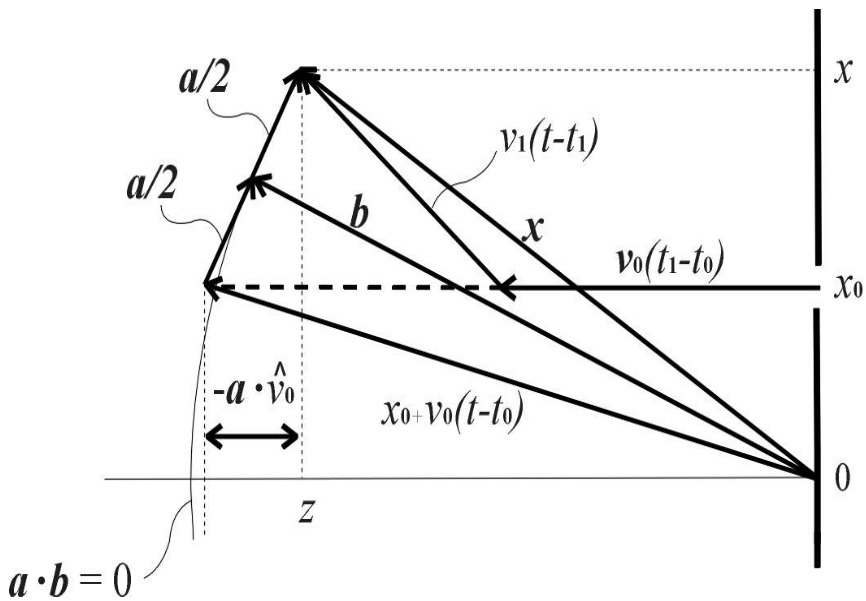

Figure 1. Definitions of Variables in Double-Slit Experiment

$$
\begin{aligned}
0 &= (1 - v_1^2/c^2)\big(c(t-t_1)\big)^2\, 2\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon}) \\
&= \left\{\left(1-\frac{v_0^2}{c^2}\right)\big(c(t-t_1)\big)^2 - 2\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\big(c(t-t_1)\big) - a^2\right\} 2\boldsymbol{a}\cdot\boldsymbol{b} \\
&\quad +2\left\{\boldsymbol{a}\cdot\left(\boldsymbol{a}+\frac{\boldsymbol{v}_0}{c}c(t-t_1)\right)\right\}\left\{\boldsymbol{x}\cdot\left(\boldsymbol{a}+\frac{\boldsymbol{v}_0}{c}c(t-t_1)\right)\right\} \\
&= \big(c(t-t_1)\big)^2\left\{\left(1-\frac{v_0^2}{c^2}\right)2\boldsymbol{a}\cdot\boldsymbol{b} + 2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right)\right\} \\
&\quad +\big(c(t-t_1)\big)\left\{-2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)2\boldsymbol{a}\cdot\boldsymbol{b} + 2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)\boldsymbol{x}\cdot\boldsymbol{a} + 2a^2\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right)\right\} \\
&\quad -a^2\, 2\boldsymbol{a}\cdot\boldsymbol{b} + 2a^2\boldsymbol{x}\cdot\boldsymbol{a}
\end{aligned}
$$

Substituting the value for t-t_1 that corresponds to exact conservation of energy, we are led to the following equation

$$0=\left[-\frac{a^2}{2\boldsymbol{a}\cdot\boldsymbol{v}_0/c}\right]^2\left\{\left(1-\frac{v_0^2}{c^2}\right)2\boldsymbol{a}\cdot\boldsymbol{b}+2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right)\right\}$$
$$+\left[-\frac{a^2}{2\boldsymbol{a}\cdot\boldsymbol{v}_0/c}\right]\left\{-2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)2\boldsymbol{a}\cdot\boldsymbol{b}+2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)\boldsymbol{x}\cdot\boldsymbol{a}+2a^2\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right)\right\}$$
$$+a^4$$

On dividing through by the known result for$[c(t\text{-}t_1)]^2$, this becomes

$$0=\left\{\left(1-\frac{v_0^2}{c^2}\right)2\boldsymbol{a}\cdot\boldsymbol{b}+2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right)\right\}$$
$$+\left[-\frac{2\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{a^2}\right]\left\{-2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)2\boldsymbol{a}\cdot\boldsymbol{b}+2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)\boldsymbol{x}\cdot\boldsymbol{a}+2a^2\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right)\right\}$$
$$+a^4\left(-\frac{2\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{a^2}\right)^2$$
$$0=\left\{\left(1-\frac{v_0^2}{c^2}\right)2\boldsymbol{a}\cdot\boldsymbol{b}-2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right)\right\}+\left(-\frac{2\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{a^2}\right)^2\left\{a^4+2a^2\boldsymbol{a}\cdot\boldsymbol{b}-a^2\boldsymbol{x}\cdot\boldsymbol{a}\right\}$$
$$=\left\{\left(1-\frac{v_0^2}{c^2}\right)2\boldsymbol{a}\cdot\boldsymbol{b}-2\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)\left(\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right)\right\}+\left(-\frac{2\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{a^2}\right)^2\left\{+a^2\boldsymbol{x}\cdot\boldsymbol{a}\right\}$$

relating the components of $\boldsymbol{a}$ when energy is conserved exactly. Since we assumed x_0 is a free parameter, this condition fixes $\boldsymbol{a}\cdot\boldsymbol{v}_0$, or equivalently $v_0(t\text{-}t_0)$, as a function of x_0. In other words, it defines a curve of endpoints of trajectories for which the phase is stationary and energy is conserved. Note that when $\boldsymbol{a}$ is nearly perpendicular to $\boldsymbol{v}_0$, the resulting value of $\boldsymbol{a}\cdot\boldsymbol{b}$ becomes

$$\boldsymbol{a}\cdot\boldsymbol{b}=a^2\frac{v_0^2}{c^2}\frac{(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)v_0(t-t_0)}{\left\{\left(1-\frac{v_0^2}{c^2}\right)a^2+2(\boldsymbol{a}\cdot\boldsymbol{v}_0/c)^2\right\}}$$
$$\approx\frac{v_0^2/c^2}{1-v_0^2/c^2}(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)v_0(t-t_0)$$

where the second result follows if it can be assumed that $2(\boldsymbol{a}\cdot\boldsymbol{v}_0/c)^2$ is negligible compared to $(1-v_0^2/c^2)a^2$. Repeated use of this approximation then produces an estimate of $\boldsymbol{a}\cdot\boldsymbol{v}_0(t\text{-}t_0)$

$$\boldsymbol{a}\cdot\boldsymbol{v}_0(t-t_0)\approx(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)$$
$$\approx-\left(1-\frac{v_0^2/c^2}{1-v_0^2/c^2}\right)^{-1}\frac{x^2-x_0^2}{2}$$

The exact result is equivalent to a 4th order equation for t-t_0 or equivalently $\boldsymbol{a}\cdot\boldsymbol{v}_0/v_0$ in terms of a_x (or equivalently x_0) where v_0/c and the components of $\boldsymbol{x}$ are given parameters.

In general, we should expect $\boldsymbol{a}\cdot\boldsymbol{v}_0$ to be negative and extremely small compared to $\boldsymbol{b}\cdot\boldsymbol{v}_0$ when energy is conserved. Furthermore, $\boldsymbol{a}\cdot\boldsymbol{b}$ will also be negative with magnitude on the order of v_0^2/c^2 times x^2-x_0^2. Substituting into the solution for t_1-t_0, we find

$$\begin{aligned} v_0\left(t_1-t_0\right) &= \frac{\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{x}_0)}{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0} \\ &\approx \frac{v_0^2/c^2}{1-v_0^2/c^2}v_0\left(t-t_0\right)-\frac{a_x x_0}{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0} \\ &\approx \frac{v_0^2/c^2}{1-v_0^2/c^2}v_0\left(t-t_0\right)+\frac{x_0}{b_x}\left(1-\frac{v_0^2/c^2}{1-v_0^2/c^2}\right)\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0 \end{aligned}$$

which shows that as x_0 becomes negative enough, the interaction can occur before the particle reaches the plane of the slits and still conserve energy. In that case, of course, the particle can only reach the target if it intersects the plane through one of the slits.

2.3 Uniqueness of Exact Energy Conservation

After multiplying through by 1-v_1^2/c^2, the equation for stationary phase can be rewritten

$$\left\{1-\frac{v_0^2}{c^2}-\frac{2\boldsymbol{a}\cdot\left(\boldsymbol{b}-\boldsymbol{x}_1\right)}{\left(c(t-t_1)\right)^2}\right\}2\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})=\left(\frac{\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})-\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{x}_1)}{c(t-t_1)}\right)^2$$

Here we notice that cross products of $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})$ and $\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{x}_1)$ occur on both sides of the equation, and by collecting these terms on the right-hand side, we are left with

$$\left\{1-\frac{v_0^2}{c^2}\right\}2\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})=\left(\frac{\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})+\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{x}_1)}{c(t-t_1)}\right)^2$$

We have now transformed the equation for stationary phase into an equivalent one where the variable factor 1-v_1^2/c^2 does not occur (except in the definition of $\boldsymbol{\varepsilon}$). Next, note that since

$$2\boldsymbol{b}-\boldsymbol{x}_1=\boldsymbol{x}+\boldsymbol{v}_0(t-t_1)$$

$$\boldsymbol{b}=\boldsymbol{x}-\frac{\boldsymbol{a}}{2}$$

the transformed equation can be rewritten such that $\boldsymbol{\varepsilon}$ is associated with $\boldsymbol{x}$

$$\left(1-\frac{v_0^2}{c^2}\right)2\boldsymbol{a}\cdot(\boldsymbol{x}+\boldsymbol{\varepsilon}-\frac{\boldsymbol{a}}{2})=\left(\frac{\boldsymbol{a}\cdot(\boldsymbol{x}+\boldsymbol{\varepsilon})}{c(t-t_1)}+\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)^2$$

which gives us a quadratic equation for $\boldsymbol{a}\cdot(\boldsymbol{x}+\boldsymbol{\varepsilon})/c(t-t_1)$

$$\left(\frac{\boldsymbol{a}\cdot(\boldsymbol{x}+\boldsymbol{\varepsilon})}{c(t-t_1)}\right)^2+2\frac{\boldsymbol{a}\cdot(\boldsymbol{x}+\boldsymbol{\varepsilon})}{c(t-t_1)}\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}-\left(1-\frac{v_0^2}{c^2}\right)c(t-t_1)\right)+\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)^2+\left(1-\frac{v_0^2}{c^2}\right)a^2=0$$

The sign choice that gives an appropriately small solution is

$$\frac{\boldsymbol{a}\cdot(\boldsymbol{x}+\boldsymbol{\varepsilon})}{c(t-t_1)}=-\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}+\left(1-\frac{v_0^2}{c^2}\right)c(t-t_1)-\left\{\left(-\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}+\left(1-\frac{v_0^2}{c^2}\right)c(t-t_1)\right)^2-\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)^2-\left(1-\frac{v_0^2}{c^2}\right)a^2\right\}^{1/2}$$

However, this is equivalent to

$$\frac{\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})}{c(t-t_1)}+\frac{a^2}{2c(t-t_1)}=-\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}+\left(1-\frac{v_0^2}{c^2}\right)c(t-t_1)-\left\{\left(-\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}+\left(1-\frac{v_0^2}{c^2}\right)c(t-t_1)\right)^2-\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)^2-\left(1-\frac{v_0^2}{c^2}\right)a^2\right\}^{1/2}$$

and, noting that $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{\varepsilon})$ is required to be positive, we see that the equation for stationary phase actually has no solution when the inequality

$$\frac{a^2}{2c(t-t_1)}>-\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}+\left(1-\frac{v_0^2}{c^2}\right)c(t-t_1)-\left\{\left(-\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}+\left(1-\frac{v_0^2}{c^2}\right)c(t-t_1)\right)^2-\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)^2-\left(1-\frac{v_0^2}{c^2}\right)a^2\right\}^{1/2}$$

holds. Isolating the radical on the right-hand side of the inequality and then squaring gives

$$\left(\frac{a^2}{2c(t-t_1)}\right)^2-2\left(\frac{a^2}{2c(t-t_1)}\right)\left(-\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}+\left(1-\frac{v_0^2}{c^2}\right)c(t-t_1)\right)>-\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)^2-\left(1-\frac{v_0^2}{c^2}\right)a^2$$

By multiplying through by$[c(t-t_1)]^2$, we see this is equivalent to

$$\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)^2\left[c(t-t_1)\right]^2+\left(\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\right)a^2\left[c(t-t_1)\right]+\frac{1}{4}a^4>0$$

This inequality obviously holds for large $|c(t-t_1)|$ but would be false between the zeros of the quadratic form on the left hand side. However, these two zeros occur at the same value

$$c(t-t_1)=-\frac{a^2}{2\boldsymbol{a}\cdot\boldsymbol{v}_0/c}$$

corresponding to exact energy conservation, which is therefore a necessary condition for the phase to be stationary with respect to t_1.

2.4 Stationary Phase Condition with Transverse Momentum as the Independent Variable

Exact solutions to the equation for stationary phase are not known yet when energy is not conserved. We have tried an approximation method based on small departures from energy conservation, but unfortunately the phase does not necessarily vary rapidly with small departures from the value of t_1 at which the phase is then found to be stationary. An alternate approach will be followed in which the independent variable is changed from t_1 to k_{1x}.

Subtracting terms in the phase that are independent of t-t_1, we are left to consider

$$\hbar\phi - L_0(t-t_0) + \boldsymbol{p}_0 \cdot \boldsymbol{x}_0 = \boldsymbol{p}_1 \cdot \boldsymbol{x} + (L_1 - L_0)(t-t_1)$$

The first term can be rewritten

$$\begin{aligned} \boldsymbol{p}_1 \cdot \boldsymbol{x} &= \frac{T_1}{c}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{x}}{c\left(t-t_1\right)} + \boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right) \\ &= \hbar k_{1x}\frac{\boldsymbol{a}\cdot\boldsymbol{x}}{a_x} + \frac{T_1}{c}\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c} \end{aligned}$$

To carry out the desired change of variables, we can solve

$$\begin{aligned} \hbar k_{1x} &= p_{1x} \\ &= \frac{T_1}{c^2}v_{1x} \\ &= \frac{mca_x}{\left\{1-v_1^2/c^2\right\}^{1/2} c\left(t-t_1\right)} \\ &= \frac{mca_x}{\left\{\left(1-v_0^2/c^2\right)\left[c\left(t-t_1\right)\right]^2 - 2\boldsymbol{a}\cdot\frac{\boldsymbol{v}_0}{c}\left[c\left(t-t_1\right)\right] - a^2\right\}^{1/2}} \end{aligned}$$

for $c(t$-$t_1)$ in terms of k_{1x}. Squaring the equation above leads to a quadratic equation in $c(t$-$t_1)$ for which the possible solutions are

$$\left(1-v_0^2/c^2\right)^{1/2}\left[c\left(t-t_1\right)\right] = \frac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} \pm \left\{a^2 + \frac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2} + \left(\frac{mca_x}{\hbar k_{1x}}\right)^2\right\}^{1/2}$$

It's clear that the upper (+) sign is the one we need if the interaction occurs before the observation. Our fundamental physical quantity is a phase shift expressible as a product of an energy and a time interval. However, we may have the option, first recognized by Dirac, of choosing the minus sign and interpreting the solution moving backwards in time as an antiparticle. For the problem at hand, though, we can use

$$\begin{aligned}
\frac{T_1}{c} &= \frac{\hbar k_{1x}}{a_x}\left[c\left(t-t_1\right)\right] \\
&= \frac{\hbar k_{1x}}{a_x\left(1-v_0^2/c^2\right)^{1/2}}\left[\frac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}+\left\{a^2+\frac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}+\left(\frac{mca_x}{\hbar k_{1x}}\right)^2\right\}^{1/2}\right] \\
&= \frac{\hbar k_{1x}\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{a_x\left(1-v_0^2/c^2\right)}+\frac{mc}{\left(1-v_0^2/c^2\right)^{1/2}}\left\{1+\left(a^2+\frac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}\right)\left(\frac{\hbar k_{1x}}{mca_x}\right)^2\right\}^{1/2}
\end{aligned}$$

to evaluate the second term in $\boldsymbol{p}_1\cdot\boldsymbol{x}$. Adding it all up, we find

$$\begin{aligned}
\hbar\phi - L_0(t-t_0)+\boldsymbol{p}_0\cdot\boldsymbol{x}_0 &= \hbar k_{1x}\frac{\boldsymbol{a}\cdot\boldsymbol{x}}{a_x}+\frac{T_1}{c}\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}+(L_1-L_0)(t-t_1) \\
&= \hbar k_{1x}\frac{\boldsymbol{a}\cdot\boldsymbol{x}}{a_x}+\frac{T_1}{c}\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}+mc\left(1-v_1^2/c^2\right)^{1/2}c(t-t_1)\left\{1-\frac{T_1}{mc^2}\left(1-v_0^2/c^2\right)^{1/2}\right\} \\
&= \hbar k_{1x}\frac{\boldsymbol{a}\cdot\boldsymbol{x}}{a_x}+\frac{T_1}{c}\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}+mc\frac{mca_x}{\hbar k_{1x}}\left\{1-\frac{T_1}{mc^2}\left(1-v_0^2/c^2\right)^{1/2}\right\} \\
&= mc\frac{mca_x}{\hbar k_{1x}}+mc\frac{\hbar k_{1x}}{mca_x}\boldsymbol{a}\cdot\boldsymbol{x}+\left\{\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}-\frac{mca_x}{\hbar k_{1x}}\left(1-v_0^2/c^2\right)^{1/2}\right\}\frac{T_1}{c}
\end{aligned}$$

Separating out the radical term in the expression above for T_1/c,

$$\begin{aligned}
&\hbar\phi - L_0(t-t_0)+\boldsymbol{p}_0\cdot\boldsymbol{x}_0 \\
&= mc\frac{mca_x}{\hbar k_{1x}}-mc\frac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}+mc\frac{\hbar k_{1x}}{mca_x}\left\{\boldsymbol{a}\cdot\boldsymbol{x}+\frac{\left(\boldsymbol{x}\cdot\boldsymbol{v}_0/c\right)\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)}{\left(1-v_0^2/c^2\right)}\right\} \\
&+\left\{\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}-\frac{mca_x}{\hbar k_{1x}}\left(1-v_0^2/c^2\right)^{1/2}\right\}\left\{\frac{T_1}{c}-\frac{\hbar k_{1x}\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{a_x\left(1-v_0^2/c^2\right)}\right\}
\end{aligned}$$

Finally, including the radical term explicitly,and choosing the + sign to get positive total energy, the phase can be written

$$
\begin{aligned}
&\hbar\phi - L_0(t-t_0) + \boldsymbol{p}_0\cdot\boldsymbol{x}_0 \\
&= mc\frac{mca_x}{\hbar k_{1x}} - mc\frac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} + mc\frac{\hbar k_{1x}}{mca_x}\left(\boldsymbol{a}\cdot\boldsymbol{x} + \frac{\left(\boldsymbol{x}\cdot\boldsymbol{v}_0/c\right)\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)}{1-v_0^2/c^2}\right) \\
&+\frac{mc}{\left(1-v_0^2/c^2\right)^{1/2}}\left\{\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c} - \frac{mca_x}{\hbar k_{1x}}\left(1-v_0^2/c^2\right)^{1/2}\right\}\left\{1+\left(a^2+\frac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}\right)\left(\frac{\hbar k_{1x}}{mca_x}\right)^2\right\}^{1/2} \\
&= mc\frac{\left(\boldsymbol{x}-\boldsymbol{a}\right)\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} + mc\frac{\hbar k_{1x}}{mca_x}\left(\boldsymbol{a}\cdot\boldsymbol{x} + \frac{\left(\boldsymbol{x}\cdot\boldsymbol{v}_0/c\right)\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)}{1-v_0^2/c^2}\right) \\
&+\frac{mc}{\left(1-v_0^2/c^2\right)^{1/2}}\left\{\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c} - \frac{mca_x}{\hbar k_{1x}}\left(1-v_0^2/c^2\right)^{1/2}\right\}\left\{\left[1+\left(a^2+\frac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}\right)\left(\frac{\hbar k_{1x}}{mca_x}\right)^2\right]^{1/2} - 1\right\}
\end{aligned}
$$

This expression is well behaved at low momentum transfer ($\hbar k_{1x} << mca_x/a$), and to second order in k_{1x} gives

$$
\begin{aligned}
&\hbar\phi - L_0(t-t_0) + \boldsymbol{p}_0\cdot\boldsymbol{x}_0 \\
&\approx mc\frac{\left(\boldsymbol{x}-\boldsymbol{a}\right)\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} + mc\left[\boldsymbol{a}\cdot\boldsymbol{x} + \frac{\left(\boldsymbol{x}\cdot\boldsymbol{v}_0/c\right)\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)}{1-v_0^2/c^2} - \frac{1}{2}\left(a^2+\frac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}\right)\right]\frac{\hbar k_{1x}}{mca_x} \\
&+\frac{mc}{\left(1-v_0^2/c^2\right)^{1/2}}\left\{\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right\}\frac{1}{2}\left(a^2+\frac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}\right)\left(\frac{\hbar k_{1x}}{mca_x}\right)^2 \\
&\approx mc\frac{\left(\boldsymbol{x}-\boldsymbol{a}\right)\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} + mc\left[\frac{\left(1-v_0^2/c^2\right)\boldsymbol{a}\cdot\boldsymbol{b} + \left(\boldsymbol{b}\cdot\boldsymbol{v}_0/c\right)\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)}{\left(1-v_0^2/c^2\right)a_x}\right]\frac{\hbar k_{1x}}{mc} \\
&+\frac{mc}{\left(1-v_0^2/c^2\right)^{1/2}}\left\{\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\right\}\frac{1}{2}\left(1+\left(\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{\left(1-v_0^2/c^2\right)^{1/2}a_x}\right)^2\right)\left(\frac{\hbar k_{1x}}{mc}\right)^2
\end{aligned}
$$

On the other hand, at high momentum transfer ($\hbar k_{1x} >> mca_x/a$), the phase can be approximated by a power series in $mc/\hbar k_{1x}$

$$
\begin{aligned}
&\hbar\phi - L_0(t-t_0) + \boldsymbol{p}_0\cdot\boldsymbol{x}_0\\
&= mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} + mc\frac{\hbar k_{1x}}{mca_x}\left(\boldsymbol{a}\cdot\boldsymbol{x} + \frac{(\boldsymbol{x}\cdot\boldsymbol{v}_0/c)(\boldsymbol{a}\cdot\boldsymbol{v}_0/c)}{1-v_0^2/c^2}\right)\\
&+\frac{mc}{\left(1-v_0^2/c^2\right)^{1/2}}\left\{\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c} - \frac{mca_x}{\hbar k_{1x}}\left(1-v_0^2/c^2\right)^{1/2}\right\}\left\{\frac{\hbar k_{1x}}{mc}\left[1+\left(\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{\left(1-v_0^2/c^2\right)^{1/2}a_x}\right)^2 + \left(\frac{mc}{\hbar k_{1x}}\right)^2\right]^{1/2} - 1\right\}\\
&\approx mc\left(\boldsymbol{a}\cdot\boldsymbol{x} + \frac{(\boldsymbol{x}\cdot\boldsymbol{v}_0/c)(\boldsymbol{a}\cdot\boldsymbol{v}_0/c)}{1-v_0^2/c^2}\right)\frac{\hbar k_{1x}}{mca_x} + \frac{mc}{\left(1-v_0^2/c^2\right)^{1/2}}\boldsymbol{x}\cdot\frac{\boldsymbol{v}_0}{c}\left[1+\left(\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{\left(1-v_0^2/c^2\right)^{1/2}a_x}\right)^2\right]^{1/2}\frac{\hbar k_{1x}}{mc}\\
&+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} - mca_x\left[1+\left(\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{\left(1-v_0^2/c^2\right)^{1/2}a_x}\right)^2\right]+\ldots
\end{aligned}
$$

In this limit, the phase has a component that is linear in k_{1x} with a large coefficient on the order of $\boldsymbol{x}\cdot\boldsymbol{v}_0/c$.

2.5 Integration over Transverse Momentum Instead of Interaction Time

We will now carry through with the change of variable in the expression for the complex probability density in the single-quantum exchange approximation

$$
\begin{aligned}
&\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)\\
&\approx -i(m/2\pi\hbar^3)\delta(y-y_0)\sum_s \int_{t_{1\min}(x_0,s)}^{t_{1\max}(x_0,s)} dt_1 \frac{V(k_{1x})e^{i\phi}}{(t-t_1)^2(1-v_1^2/c^2)^2}
\end{aligned}
$$

from t_1 to k_{1x}. We can do this by differentiating p_{1x}, which is a function of t-t_1 and at the same time equal to $\hbar k_{1x}$

$$\frac{dp_{1x}}{dt_1} = \frac{T_1}{c^2}\frac{dv_{1x}}{dt_1} + \frac{T_1}{c^2(1-v_1^2/c^2)}\left(\frac{\boldsymbol{v}_1}{c}\cdot\frac{d\boldsymbol{v}_1}{cdt_1}\right)v_{1x}$$

$$= \frac{T_1}{c^2}\frac{dv_{1x}}{dt_1} + \frac{T_1}{c^2(1-v_1^2/c^2)}\left(\frac{a^2}{\left[c\left(t-t_1\right)\right]^2} + \frac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{c\left(t-t_1\right)}\right)\frac{v_{1x}}{t-t_1}$$

$$= \frac{T_1}{c^2\left(1-v_1^2/c^2\right)}\left(1-v_1^2/c^2 + \frac{a^2}{\left[c\left(t-t_1\right)\right]^2} + \frac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{c\left(t-t_1\right)}\right)\frac{v_{1x}}{t-t_1}$$

$$= \frac{T_1}{c^2\left(1-v_1^2/c^2\right)}\left(1-v_0^2/c^2 - \frac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{c\left(t-t_1\right)}\right)\frac{v_{1x}}{t-t_1}$$

The combination of factors in the integral that we need therefore becomes

$$\frac{dt_1}{\left(t-t_1\right)^2(1-v_1^2/c^2)^2} = \frac{\hbar c^2 dk_{1x}}{a_x\left(1-v_1^2/c^2\right)T_1}\frac{1}{\left(1-v_0^2/c^2 - \dfrac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{c\left(t-t_1\right)}\right)}$$

$$= \frac{\hbar dk_{1x}}{ma_x\left(1-v_1^2/c^2\right)^{1/2}}\frac{c\left(t-t_1\right)}{\left(1-v_0^2/c^2\right)^{1/2}\left(\left(1-v_0^2/c^2\right)^{1/2}c\left(t-t_1\right) - \dfrac{\boldsymbol{a}\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}$$

$$= \frac{\hbar dk_{1x}}{ma_x\left(1-v_1^2/c^2\right)}\frac{mca_x}{\left(1-v_0^2/c^2\right)^{1/2}\hbar k_{1x}\left\{a^2 + \dfrac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2} + \left(\dfrac{mca_x}{\hbar k_{1x}}\right)^2\right\}^{1/2}}$$

$$= \frac{\hbar dk_{1x}}{\left(1-v_0^2/c^2\right)^{1/2}\left(1-v_1^2/c^2\right)ma_x\left\{1+\left(a^2 + \dfrac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}\right)\left(\dfrac{\hbar k_{1x}}{mca_x}\right)^2\right\}^{1/2}}$$

Thus after the change of variables, the complex probability density becomes

$$\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$
$$= -i(m/2\pi\hbar^3)\delta(y-y_0)\sum_s\int_{k_{1x\min}(x_0,s)}^{k_{1x\max}(x_0,s)} dk_{1x}\frac{\hbar V(k_{1x})e^{i\phi}}{\left(1-v_0^2/c^2\right)^{1/2}\left(1-v_1^2/c^2\right)ma_x\left\{1+\left(a^2 + \dfrac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}\right)\left(\dfrac{\hbar k_{1x}}{mca_x}\right)^2\right\}^{1/2}}$$

This can be useful because the limits of the integration over k_{1x} depend on whether the initial trajectory passes through the slit in question and are otherwise almost fixed values (for each slit).

2.5.1 Integration over Transverse Momentum on Blocked Trajectories

Referring to Figure 2, we see that the lower limit of k_{1x} (as a function of x_0) corresponds to the earliest interaction time $t_{1\min}$ where

$$\frac{x - x_{0\max}}{v_0(t-t_0) + \boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0} = \frac{x - x_0}{v_0(t-t_{1\min}) + \boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}$$

This results in the particle intersecting the plane at the top ($x_{0\max}$) of slit s. Solving for $v_0(t\text{-}t_{1\min})$ gives

$$\begin{aligned} v_0(t-t_{1\min}) &= \frac{x-x_0}{x-x_{0\max}} v_0\left(t-t_0\right) + \frac{x_{0\max}-x_0}{x-x_{0\max}} \boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 \\ &= \frac{x-x_0}{x-x_{0\max}}\left(v_0\left(t-t_0\right) + \boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right) - \boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 \\ &= \left(\frac{x-x_0}{x-x_{0\max}}\right)\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0 - \boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 \end{aligned}$$

Assuming the initial trajectory does not pass through the slit, a similar relation exists between the maximum value of t_1 and the position of the bottom of the slit

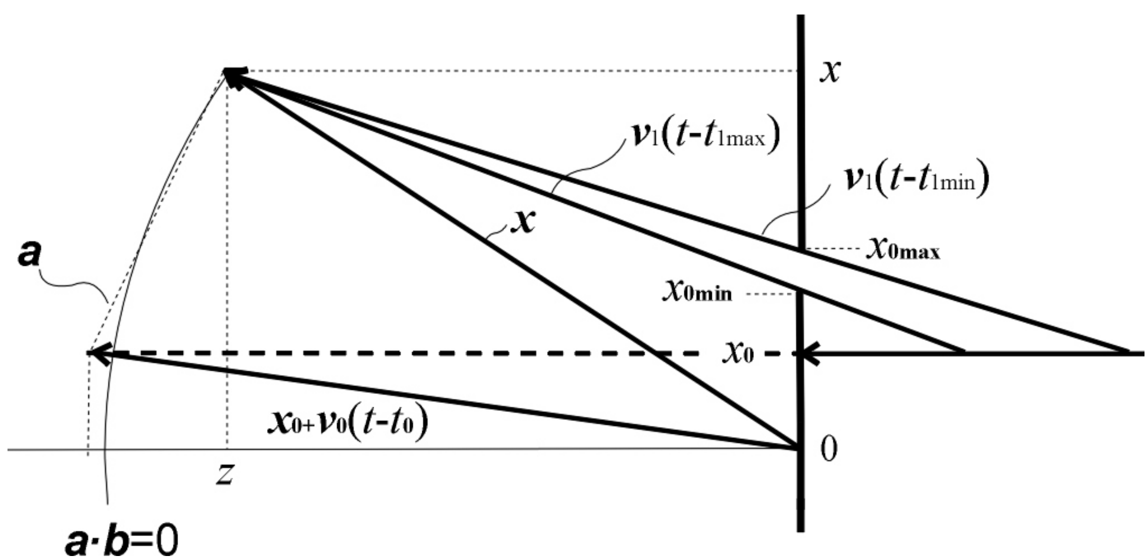

Figure 2. Definitions of Minimum and Maximum Interaction Times

$$
\begin{aligned}
v_0(t-t_{1\max}) &= \frac{x-x_0}{x-x_{0\min}}\left(v_0\left(t-t_0\right)+\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)-\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 \\
&= \left(\frac{x-x_0}{x-x_{0\min}}\right)\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0-\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0
\end{aligned}
$$

It therefore turns out that when $\boldsymbol{a}\cdot\boldsymbol{v}_0$ is small compared to $\boldsymbol{x}\cdot\boldsymbol{v}_0$ (we shall see that it is very small), the x-components of the velocity at the two limits are small and approximately independent of x_0.

$$
\begin{aligned}
v_{1x\max} &= \frac{\left(x-x_0\right)v_0}{\left(\frac{x-x_0}{x-x_{0\min}}\right)\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0-\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0} \\
&= \frac{\left(x-x_{0\min}\right)v_0}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0-\frac{x-x_{0\min}}{x-x_0}\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0} \\
&\approx \frac{\left(x-x_{0\min}\right)v_0}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0} \\
v_{1x\min} &= \frac{\left(x-x_{0\max}\right)v_0}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0-\frac{x-x_{0\max}}{x-x_0}\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0} \\
&\approx \frac{\left(x-x_{0\max}\right)v_0}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}
\end{aligned}
$$

These velocities are, however, slightly different for each slit. In the same approximation, the component of the velocity along the initial direction remains essentially equal to v_0. This means that the momentum transfer is small and nearly constant at

$$
\begin{aligned}
\hbar\overline{k}_{1xs} &= m\overline{v}_{1xs} \\
&\approx \frac{x-\left(x_{0\min}+x_{0\max}\right)/2}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}mv_0 = \frac{x-\overline{x}_{0s}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}mv_0
\end{aligned}
$$

It will be noticed that the difference in mean momentum for the two slits is small in proportion to the distance from the axis of symmetry to slit position compared to the target position

$$
\hbar\overline{k}_{1xs} \approx \hbar\overline{k}_{1x}\left(1-\frac{\overline{x}_{0s}}{x}\right)
$$

where k_{1x}-bar corresponds to scattering on the axis of symmetry in the plane of the slits. The width of the range of the integration will be

$$\begin{aligned}\hbar\delta k_{1x} &= m\left(v_{1x\max} - v_{1x\min}\right) \\ &= \frac{x_{0\max} - x_{0\min}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0} m v_0 \\ &= \frac{w}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0} m v_0\end{aligned}$$

and its magnitude compared to the mean momentum transfer

$$\frac{\delta k_{1x}}{\bar{k}_{1xs}} \approx \frac{w}{x - \bar{x}_{0s}}$$

will also be small if the slit geometry is compressed in order to make the first minimum occur at a larger value of x where it should be easier to observe.

Of course, the limiting and mean velocities are slightly different for each slit, so the complex probability density becomes approximately

$$\begin{aligned}&\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)\Big|_{|\boldsymbol{a}\cdot\boldsymbol{v}_0|\ll \boldsymbol{x}\cdot\boldsymbol{v}_0} \\ &\approx -i(1/2\pi\hbar^2)\delta(y-y_0)\sum_s V(\bar{k}_{1xs})\int_{k_{1x\min}(x_0,s)}^{k_{1x\max}(x_0,s)} dk_{1x}\frac{e^{i\phi}}{\left(1-v_0^2/c^2\right)^{1/2}\left(1-v_1^2/c^2\right)a_x\left\{1+\left(a^2+\dfrac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2}\right)\left(\dfrac{\hbar k_{1x}}{mca_x}\right)^2\right\}^{1/2}}\end{aligned}$$

In a typical case, the mean values of momentum transfer for the two slits are not much different, so we get the expected form that is proportional to the Fourier transform at the wave vector that corresponds to scattering in the vicinity of the slits.

To second order in $\hbar k_{1x}/mc$ and $\boldsymbol{a}\cdot\boldsymbol{v}_0/a_x$, but excluding terms of second order in both at the same time, the phase can be written

$$\begin{aligned}\hbar\phi - L_0(t-t_0) + \boldsymbol{p}_0\cdot\boldsymbol{x}_0 &\approx mc\frac{\left(\boldsymbol{x}-\boldsymbol{a}\right)\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} + \hbar\left\{k_{1x}\xi + \sigma^2 k_{1x}^2\right\} \\ &\approx mc\frac{\left(\boldsymbol{x}-\boldsymbol{a}\right)\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}} - \hbar\left(\frac{\xi}{\sigma}\right)^2 + \hbar\left(\frac{\xi}{\sigma}+\sigma k_{1x}\right)^2\end{aligned}$$

where we have introduced the definitions

$$\xi \equiv \frac{\left(1 - v_0^2/c^2\right)\boldsymbol{a}\cdot\boldsymbol{b} + \left(\boldsymbol{b}\cdot\boldsymbol{v}_0/c\right)\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)}{\left(1 - v_0^2/c^2\right)a_x}$$

$$\sigma = \left(\frac{\boldsymbol{x}\cdot\boldsymbol{v}_0/c}{2\left(1 - v_0^2/c^2\right)^{1/2}}\frac{\hbar}{mc}\right)^{1/2}$$

Note that, to the extent that $\boldsymbol{a}\cdot\boldsymbol{v}_0$ is small compared to $a_x v_0$, the coefficient of the quadratic term in k_{1x} is approximately independent of x_0. The phase is stationary centered at momentum transfer

$$\begin{aligned}\frac{\hbar k_{1x}^{stationary}}{mc} &\approx -\frac{\hbar}{mc}\frac{\xi}{\sigma^2} \\ &\approx -2\left(1 - v_0^2/c^2\right)^{1/2}\frac{\xi}{\boldsymbol{x}\cdot\boldsymbol{v}_0/c}\end{aligned}$$

The linear term in the phase becomes significant when $k_{1x}\xi$ is of the order of unity or larger. In a non-relativistic experiment, the quadratic term in the phase is likely to be small on blocked trajectories, as can be seen by rearranging

$$\begin{aligned}\left(\sigma\overline{k}_{1x}\right)^2 &= \left(\frac{\boldsymbol{x}\cdot\boldsymbol{v}_0/c}{2\left(1 - v_0^2/c^2\right)^{1/2}}\frac{\hbar}{mc}\right)\left(\frac{x - \overline{x}_{0s}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\frac{mv_0}{\hbar}\right)^2 \\ &= \left(\frac{1}{2\left(1 - v_0^2/c^2\right)^{1/2}}\frac{x - \overline{x}_{0s}}{\overline{x}_{0s}}\frac{v_0^2}{c^2}\right)\left(\frac{mv_0}{\hbar}\frac{x - \overline{x}_{0s}}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\overline{x}_{0s}\right)\end{aligned}$$

The second factor would be about $\pm\pi/4$ if x corresponds to the first minimum at the target. Supposing we have chosen x to be about 10 times x_{0s}-bar, in order to make the first minimum more easily observable, the first factor will be less than 10 $(v_0/c)^2$ and would be small if v_0/c is less than about 10^{-1}.

The phase will vary rapidly at values of momentum transfer where relativistic effects are significant, that is if $\hbar k_{1x}/mca_x$ is not negligible compared to unity. However, the integrand remains bounded because the Fourier transform of the potential contributes a factor of k_{1x} in the denominator

$$\frac{V(k_{1x})}{\left(1-v_1^2/c^2\right)a_x\left\{1+\left(a^2+\dfrac{(\boldsymbol{a}\cdot\boldsymbol{v}_0/c)^2}{1-v_0^2/c^2}\right)\left(\dfrac{\hbar k_{1x}}{mca_x}\right)^2\right\}^{1/2}}\sim\left(\frac{k_{1x}V(k_{1x})}{\left(1-v_1^2/c^2\right)\left(\dfrac{\hbar k_{1x}}{mc}\right)^2}\right)\frac{\hbar a^{-1}}{mc}$$
$$\sim\left(\frac{k_{1x}V(k_{1x})}{v_{1x}^2/c^2}\right)\frac{\hbar a^{-1}}{mc}$$

which along with the factor that was already there, combines with the Lorentz factor ($1-v_1^2/c^2$) to give v_{1x}^2/c^2 which approaches unity in this limit. Thus, for the blocked trajectories, the complex probability density becomes approximately

$$\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)\Big|_{|\boldsymbol{a}\cdot\boldsymbol{v}_0|\ll\boldsymbol{x}\cdot\boldsymbol{v}_0}$$
$$\approx -i(1/2\pi\hbar^2)\delta(y-y_0)\sum_s\frac{V(\bar{k}_{1xs})}{\left(1-v_0^2/c^2\right)^{3/2}a_x}\int_{k_{1x\min}(x_0,s)}^{k_{1x\max}(x_0,s)}dk_{1x}e^{i\phi}$$
$$\approx -i(1/2\pi\hbar^2)\delta(y-y_0)\sum_s\frac{V(\bar{k}_{1xs})e^{(i/\hbar)\left(L_0(t-t_0)-\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}}\frac{1}{a_x}\int_{k_{1x\min}(x_0,s)}^{k_{1x\max}(x_0,s)}dk_{1x}e^{ik_{1x}\left\{\xi+\frac{\boldsymbol{x}\cdot\boldsymbol{v}_0/c}{2\left(1-v_0^2/c^2\right)^{1/2}}\left(\frac{\hbar k_{1x}}{mc}\right)\right\}}$$

Note that the contribution from trajectories passing though each slit s are separately proportional to $V(k_{1xs}$-bar), and each shows the minimum that is attributed to destructive interference in conventional quantum mechanics. In typical cases, the transform of the potential is almost the same for trajectories passing through the two slits, so their contributions do not necessarily cancel or "interfere."

Noting that when the quadratic term is negligible over the restricted range of momentum transfer on blocked trajectories, we can evaluate the integral over dk_{1x}

$$\int_{k_{1x\min}(x_0,s)}^{k_{1x\max}(x_0,s)}dk_{1x}e^{ik_{1x}\left\{\xi+\frac{\boldsymbol{x}\cdot\boldsymbol{v}_0/c}{2\left(1-v_0^2/c^2\right)^{1/2}}\left(\frac{\hbar k_{1x}}{mc}\right)\right\}}=\int_{k_{1x\min}(x_0,s)}^{k_{1x\max}(x_0,s)}dk_{1x}e^{i\left\{k_{1x}\xi+\sigma^2k_{1x}^2\right\}}$$
$$\approx\int_{k_{1x\min}(x_0,s)}^{k_{1x\max}(x_0,s)}dk_{1x}e^{ik_{1x}\xi}$$
$$\approx\frac{e^{ik_{1x\max}(x_0,s)\xi}-e^{ik_{1x\min}(x_0,s)\xi}}{i\xi}$$
$$\approx\delta k_{1x}e^{i\bar{k}_{1xs}\xi}\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}$$

2.5.2 Integration over Transverse Momentum on Unblocked Trajectories

If, on the other hand, the initial trajectory does pass through a slit, the upper limit of the integration over k_{1x} becomes infinite. Relevant to that limit, the Fourier transform of the potential

$$\begin{aligned}
V(k_{1x}) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx' V(x') e^{-ik_{1x}x'} \\
&= V_0 \delta\left(k_{1x}\right) - \frac{V_0}{2\pi}\int_{slits} dx' e^{-ik_{1x}x'} \\
&= V_0 \delta\left(k_{1x}\right) - \frac{V_0}{2\pi}\left[\frac{e^{-ik_{1x}x'}}{-ik_{1x}} \Bigg|_{-L/2-W}^{-L/2} + \Bigg|_{L/2}^{L/2+W} \right. \\
&= V_0 \delta\left(k_{1x}\right) + \frac{V_0}{k_{1x}\pi}\left\{ \sin k_{1x}\left(L/2+W\right) - \sin k_{1x}\left(L/2\right) \right\}
\end{aligned}$$

contributes another factor of k_{1x} in the denominator. We have assumed that the delta function is outside the range of the integration over k_{1x}, so we have

$$\begin{aligned}
&\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0) \\
&= -i(m/2\pi\hbar^3)\delta(y-y_0)\sum_s \int_{k_{1x\min}(x_0,s)}^{\infty} dk_{1x} \frac{\hbar V_0 \left\{ \sin k_{1x}\left(L/2+W\right) - \sin k_{1x}\left(L/2\right) \right\} e^{i\phi}}{\pi m a_x \left(1-v_0^2/c^2\right)^{1/2}\left(1-v_1^2/c^2\right) k_{1x} \left\{ 1 + \left(a^2 + \dfrac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2} \right) \left(\dfrac{\hbar k_{1x}}{mca_x} \right)^2 \right\}^{1/2}}
\end{aligned}$$

At high momentum transfer, the denominator of the integrand scales like $k_{1x}^2(1-v_1^2/c^2)=(mv_{1x}/\hbar)^2$, which approaches a constant value because v_{1x} is bounded by c. Also in this limit, the phase is again approximately linear in k_{1x}, but its coefficient is on the order of $-\boldsymbol{x}\cdot\boldsymbol{v}_0/c$, which is large in magnitude compared to $L/2+W$ and $L/2$ in the cases of primary interest. Therefore, the integrand will oscillate rapidly and contribute little to the complex probability density.

In typical cases, $\hbar k_{1x\min}/mc$ is on the order of

$$\frac{\hbar k_{1x\min}}{mc} \approx \frac{\left(x - x_{0\max}\right)}{\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\frac{v_0}{c}$$

assuming that $\boldsymbol{a}\cdot\boldsymbol{v}_0$ is small compared to $\boldsymbol{x}\cdot\boldsymbol{v}_0$. On the other hand, because we are interested in the first null in the diffraction pattern,

$$\frac{\sin k_{1x}\left(L/2+W\right)-\sin k_{1x}\left(L/2\right)}{k_{1x}}=\frac{\sin k_{1x}\left(\frac{L+W}{2}+\frac{W}{2}\right)-\sin k_{1x}\left(\frac{L+W}{2}-\frac{W}{2}\right)}{k_{1x}}$$
$$=\frac{2\cos k_{1x}\frac{L+W}{2}\sin k_{1x}\frac{W}{2}}{k_{1x}}$$

it follows that $k_{1\text{xmin}}(L+W)/2$ will be of the order of unity, or more specifically $\pi/2$. Therefore, when the relativistic corrections become significant, $k_{1\text{x}}$ will be many orders of magnitude larger than $k_{1\text{xmin}}$, and the integrand will be many orders of magnitude smaller. Therefore, oscillations at a small but finite amplitude at high momentum transfer can be ignored in a first approximation and we can consider

$$\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$
$$\approx -i(1/2\pi\hbar^2)\delta(y-y_0)\sum_s \frac{V_0}{\pi a_x\left(1-v_0^2/c^2\right)^{3/2}}\int_{k_{1x\min}(x_0,s)}^{\infty} dk_{1x}\frac{\left\{\sin k_{1x}\left(L/2+W\right)-\sin k_{1x}\left(L/2\right)\right\}e^{i\phi}}{k_{1x}}$$
$$\approx -i(/2\pi\hbar^2)\delta(y-y_0)\sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)\,\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{a_x\left(1-v_0^2/c^2\right)^{3/2}}\frac{V_0}{\pi}\int_{k_{1x\min}(x_0,s)}^{\infty}\frac{dk_{1x}}{k_{1x}}\left\{\sin k_{1x}\left(L/2+W\right)-\sin k_{1x}\left(L/2\right)\right\}e^{i\left\{k_{1x}\xi+\sigma^2k_{1x}^2\right\}}$$

It is useful to extend the lower limit of the integration to 0 and then consider the compensating integral over the finite limits separately

$$\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$
$$\approx -i(1/2\pi\hbar^2)\delta(y-y_0)\sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)\,\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{a_x\left(1-v_0^2/c^2\right)^{3/2}}\frac{V_0}{\pi}\left\{\begin{array}{l}\displaystyle\int_0^{\infty}\frac{dk_{1x}}{k_{1x}}\left\{\sin k_{1x}\left(L/2+W\right)-\sin k_{1x}\left(L/2\right)\right\}\begin{pmatrix}\cos k_{1x}\xi\\ +i\sin k_{1x}\xi\end{pmatrix}e^{i\sigma^2k_{1x}^2}\\ \displaystyle-\int_0^{k_{1x\min}(x_0,s)}\frac{dk_{1x}}{k_{1x}}2\cos k_{1x}\frac{L+W}{2}\sin k_{1x}\frac{W}{2}e^{i\left\{k_{1x}\xi+\sigma^2k_{1x}^2\right\}}\end{array}\right\}$$
$$\approx -i(1/2\pi\hbar^2)\delta(y-y_0)\sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)\,\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{a_x\left(1-v_0^2/c^2\right)^{3/2}}\frac{V_0}{\pi}\left\{\begin{array}{l}\displaystyle\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk_{1x}}{k_{1x}}\left\{\sin k_{1x}\left(L/2+W\right)-\sin k_{1x}\left(L/2\right)\right\}e^{i\left\{k_{1x}\xi+\sigma^2k_{1x}^2\right\}}\\ \displaystyle+i\int_0^{\infty}\frac{dk_{1x}}{k_{1x}}\left\{\sin k_{1x}\left(L/2+W\right)-\sin k_{1x}\left(L/2\right)\right\}\left(\sin k_{1x}\xi\right)e^{i\sigma^2k_{1x}^2}\\ \displaystyle-\int_0^{k_{1x\min}(x_0,s)}\frac{dk_{1x}}{k_{1x}}2\cos k_{1x}\frac{L+W}{2}\sin k_{1x}\frac{W}{2}e^{i\left\{k_{1x}\xi+\sigma^2k_{1x}^2\right\}}\end{array}\right\}$$

which is possible because the integrand is well behaved at $k_{1\text{x}}=0$. The real part of the infinite integral reproduces the potential evaluated at ξ with some loss of high spatial frequencies due to the quadratic term in the phase. If this loss can be neglected, that is σ can be set to zero, we then have

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk_{1x}}{k_{1x}}\left\{\sin k_{1x}\left(L/2+W\right)-\sin k_{1x}\left(L/2\right)\right\}e^{i\left\{k_{1x}\xi+\sigma^2 k_{1x}^2\right\}}\approx-\frac{\pi}{2}\left\{\begin{matrix}-\frac{L}{2}-W<\xi<-\frac{L}{2}\\ \frac{L}{2}<\xi<\frac{L}{2}+W\end{matrix}\right\}$$

In other words, this part of the result is a non-zero constant when ξ, which has the dimension of length, is "in" one of the slits. If we set σ equal to 0 first, the remaining imaginary part of the infinite integral is

$$i\int_{0}^{\infty}\frac{dk_{1x}}{k_{1x}}\left\{\sin k_{1x}\left(L/2+W\right)-\sin k_{1x}\left(L/2\right)\right\}\left(\sin k_{1x}\xi\right)=\frac{i}{4}\ln\left(\frac{\left(L/2+W+\xi\right)\left(L/2-\xi\right)}{\left(L/2+W-\xi\right)\left(L/2+\xi\right)}\right)^2$$

according to Gradshteyn and Ryzhik [5] 3.741 #1 (page 414). This result has logarithmic singularities where ξ coincides with the edges of the slits, but it becomes rapidly smaller as ξ moves away from the slits. For the time being, it will be assumed that this form gives correct results after the integration over dx_0.

Finally, if the quadratic term in the phase can be neglected in the infinite integrals over dk_{1x}, it will be reasonable to neglect it in the integral from 0 to k_{1xmins} as well

$$\rho_1(\boldsymbol{x},t;\boldsymbol{x}_0,t_0,\boldsymbol{v}_0)$$

$$\approx -i(1/2\pi\hbar^2)\delta(y-y_0)\sum_s\frac{e^{(i/\hbar)\left(L_0(t-t_0)\,\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{a_x\left(1-v_0^2/c^2\right)^{3/2}}\frac{V_0}{\pi}\left\{\begin{matrix}-\frac{\pi}{2}\left\{\begin{matrix}-\frac{L}{2}-W<\xi<-\frac{L}{2}\\ \frac{L}{2}<\xi<\frac{L}{2}+W\end{matrix}\right\}\\ \frac{i}{4}\ln\left(\frac{\left(L/2+W+\xi\right)\left(L/2-\xi\right)}{\left(L/2+W-\xi\right)\left(L/2+\xi\right)}\right)^2\\ -\int_0^{k_{1x\min}(x_0,s)}\frac{dk_{1x}}{k_{1x}}2\cos k_{1x}\frac{L+W}{2}\sin k_{1x}\frac{W}{2}e^{ik_{1x}\xi}\end{matrix}\right\}$$

2.6 Integrating Over the Beam

So far, we have calculated the complex probability density resulting from a single particle entering with velocity $\boldsymbol{v}_0$ along the z-axis on a trajectory passing through some fixed $\boldsymbol{x}_0$. Of course, we don't know how to prepare that initial state. Most likely, we would have a distant point source, and the displacement of the initial trajectory from the z-axis would be unknown. Because of the translation invariance of the boundary conditions along the y-axis, integrating the probability density over y_0 will replace the delta function $\delta(y\text{-}y_0)$ by N_y which is the density of particles in the y-direction. Likewise, adding up the contributions in the x-direction entails multiplying by a factor of N_x as well as integrating over dx_0. The final result, of course is then independent of x_0.

2.6.1 Integrating Over the Beam on Blocked Trajectories

For the above reasons, for blocked trajectories, the averaged contribution becomes

$$\begin{aligned}
&\rho_1\left(\boldsymbol{x},t;t_0,\boldsymbol{v}_0\right) \\
&\approx -i(1/2\pi\hbar^2)N_xN_y\sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)-\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}} V(\overline{k}_{1xs})\delta k_{1x}\int_{-\infty}^{x_{0\min s}} \frac{dx_0}{a_x} e^{i\overline{k}_{1xs}\xi}\,\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}
\end{aligned}$$

The integral over dx_0 can be put in a more standard form by changing the variable of integration to ξ,, which gives

$$\begin{aligned}
d\xi &= \left(\frac{a_x}{2}+\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)a_x}\right)\frac{dx_0}{a_x} \\
&= \pm\left(\left(\xi-x\right)^2+2\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}\frac{dx_0}{a_x}
\end{aligned}$$

It also follows from the definition of ξ that the phase term $k_{1x}\xi$ of the integrand increases monotonically with x_0 over the range $(-\infty, x)$ in which a_x is positive if

$$\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{1-v_0^2/c^2}=\frac{\left(\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right)^2-v_0^2\left(t-t_0\right)^2}{2\left(1-v_0^2/c^2\right)}>0$$

Otherwise, (that is if $\boldsymbol{a}\cdot\boldsymbol{v}_0<0$) ξ passes through a maximum value ξ_{max} at

$$\begin{aligned}
\xi_{\max} &= x-\left(x-x_{0sp}\right) \\
&= x_{0sp}
\end{aligned}$$

at the value of $x_0=x_{0sp}$ (for stationary phase) given by

$$\begin{aligned}
x-x_{0sp} &= \left\{-2\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{1-v_0^2/c^2}\right\}^{1/2} \\
&= \left\{-\frac{\left(\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right)^2-v_0^2\left(t-t_0\right)^2}{1-v_0^2/c^2}\right\}^{1/2}
\end{aligned}$$

Figure 3 shows examples of $k_{1x}\xi$ plotted vs x_0/x at different values of x_{0sp}/x assuming the incoming particle is non-relativistic and x_{0s}-bar$/x=0.1$, $xv_0/\boldsymbol{x}\cdot\boldsymbol{v}_0=10^{-5}$ and $mv_0x/\hbar=1.2(10)^7$.

Solving for $\boldsymbol{a}\cdot\mathbf{v}_0$, we find

$$\left(-\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)^2+2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\left(-\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)-\left(1-v_0^2/c^2\right)\left(x-x_{0sp}\right)^2>0$$

$$-\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0>-\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0+\left(\left(\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right)^2+\left(1-v_0^2/c^2\right)\left(x-x_{0sp}\right)^2\right)^{1/2}\approx\frac{\left(1-v_0^2/c^2\right)\left(x-x_{0sp}\right)^2}{2\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}$$

Roughly, the minimum value of - $\boldsymbol{a}\cdot\boldsymbol{v}_0/v_0$ is as small compared to x-x_{0sp} as x-x_{0sp} is to $2\boldsymbol{x}\cdot\boldsymbol{v}_0/v_0$, It will be noted that exact conservation of energy on the initial trajectory defined by x_0=x_{0sp} can be in this range. Under the assumption that $(\boldsymbol{a}\cdot\boldsymbol{v}_0/c)^2$ is negligible compared to $(1\text{-}v_0^2/c^2)a^2$, energy is conserved if

$$-x_{0sp}<\left(\frac{v_0^2/c^2}{1-v_0^2/c^2}\right)x$$

Of course, the initial trajectories that contribute the most to the complex probability density also correspond to small values of ξ_{max}=$x_{0sp.}$

Upon the change of variables, the term that then appears in the denominator, instead of $a_{x,}$ upon integrating over $d\xi$ instead of dx_0 can be evaluated by solving the definition of ξ as a quadratic equation for a_x,

$$a_x=-(\xi-x)\pm\left((\xi-x)^2+2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}$$

giving

$$\frac{a_x}{2}+\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)a_x}=\pm\left((\xi-x)^2+2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}$$

For positive values of a_x the positive sign is required when $\boldsymbol{a}\cdot\boldsymbol{v}_0$ is positive but when $\boldsymbol{a}\cdot\boldsymbol{v}_0$ is negative, the negative sign is needed in the range x_{0sp}<x_0<x_{0mins} so

$$\frac{dx_0}{a_x}=\frac{x_{0sp}-x_0}{\left|x_{0sp}-x_0\right|}\frac{d\xi}{\left((\xi-x)^2+2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}}$$

and the net result of changing the variable of integration is

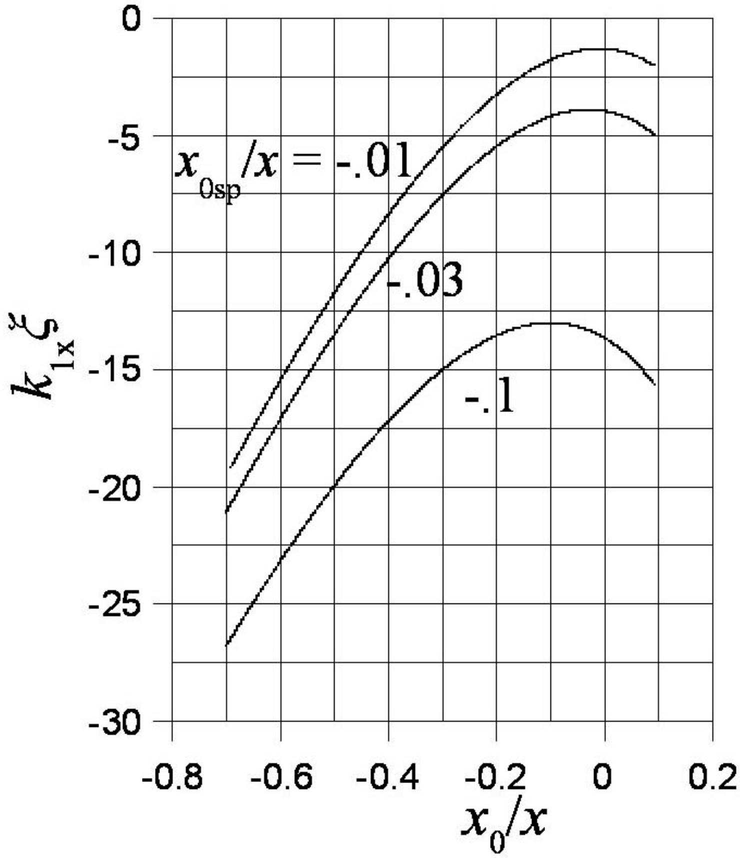

Figure 3 Non-relativistic double slit experiment with x_{0s}-bar$/x$ =0.1, $xv_0/ x \cdot v_0 = 10^{-5}$ and $mv_0x/\hbar$ = 1.2(10)^7.

$$\rho_1\left(\boldsymbol{x},t;t_0,\boldsymbol{v}_0\right)$$

$$\approx -i(1/2\pi\hbar^2)N_xN_y\sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)-\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}}V(\overline{k}_{1xs})\delta k_{1x}$$

$$\times\int_{-\infty}^{\xi(x_{0\min s})}\frac{x_{0sp}-x_0}{\left|x_{0sp}-x_0\right|}\frac{d\xi}{\left(\left(\xi-x\right)^2+2\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}}e^{i\overline{k}_{1xs}\xi}\left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}\right)$$

If x_{0sp} is in the range of the integration the integral over $d\xi$, it will be the maximum value reached by ξ, and the integral can be considered in two parts

$$\int_{-\infty}^{\xi(x_{0\min s})} \frac{x_{0sp}-x_0}{\left|x_{0sp}-x_0\right|} \frac{d\xi}{\left((\xi - x)^2 + 2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)} \right)^{1/2}} e^{i\bar{k}_{1xs}\xi} \left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2} \right)$$

$$= \int_{-\infty}^{x_{0sp}} \frac{d\xi}{\left((\xi - x)^2 + 2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)} \right)^{1/2}} e^{i\bar{k}_{1xs}\xi} \left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2} \right) + \int_{\xi(x_{0\min s})}^{x_{0sp}} \frac{d\xi}{\left((\xi - x)^2 + 2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)} \right)^{1/2}} e^{i\bar{k}_{1xs}\xi} \left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2} \right)$$

If, in addition, k_{1xs}-*bar* is much larger than δk_{1x} (making the first minimum at the target more easily observable) the phase term k_{1xs}-bar ξ will begin to oscillate rapidly before $\delta k_{1x}\,\xi/2$ becomes significantly different from $\delta k_{1xs}x_{0sp}/2$. In such cases, the integral over $d\xi$ becomes approximately

$$\int_{-\infty}^{\xi(x_{0\min s})} \frac{x_{0sp}-x_0}{\left|x_{0sp}-x_0\right|} \frac{d\xi}{\left((\xi - x)^2 + 2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)} \right)^{1/2}} e^{i\bar{k}_{1xs}\xi} \left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2} \right)$$

$$\approx \frac{\sin\delta k_{1x}x_{0sp}/2}{\delta k_{1x}x_{0sp}/2} \left\{ \int_{-\infty}^{x_{0sp}} \frac{d\xi e^{i\bar{k}_{1xs}\xi}}{\left((\xi - x)^2 + 2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)} \right)^{1/2}} + \int_{\xi(x_{0\min s})}^{x_{0sp}} \frac{d\xi e^{i\bar{k}_{1xs}\xi}}{\left((\xi - x)^2 + 2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)} \right)^{1/2}} \right\}$$

These integrals can be put in more standard forms by changing the integration variable to

$$\begin{aligned}
(\xi - x)^2 + 2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)} &= (\xi - x)^2 - \left(x - x_{0sp}\right)^2 \\
&= \left(x - x_{0sp}\right)^2 \left\{ \frac{(\xi - x)^2}{\left(x - x_{0sp}\right)^2} - 1 \right\} \\
&= \left(x - x_{0sp}\right)^2 \left\{ v^2 - 1 \right\}
\end{aligned}$$

and choosing the positive range for the variable v results in

$$\int_{-\infty}^{\xi(x_{0\min s})} \frac{x_{0sp}-x_0}{\left|x_{0sp}-x_0\right|} \frac{d\xi}{\left(\left(\xi-x\right)^2+2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}} e^{i\bar{k}_{1xs}\xi}\left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}\right)$$

$$\approx \frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}\left\{\int_{-\infty}^{x_{0sp}} \frac{d\xi e^{i\bar{k}_{1xs}\xi}}{\left(\left(\xi-x\right)^2+2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}}+\int_{\xi(x_{0\min s})}^{x_{0sp}} \frac{d\xi e^{i\bar{k}_{1xs}\xi}}{\left(\left(\xi-x\right)^2+2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}}\right\}$$

$$\approx \frac{\sin\delta k_{1x}x_{0sp}/2}{\delta k_{1x}x_{0sp}/2} e^{i\bar{k}_{1xs}x}\left\{\int_1^{\infty}\frac{d\nu e^{-i\bar{k}_{1xs}(x-x_{0sp})\nu}}{\left(\nu^2-1\right)^{1/2}}+\int_1^{(x-\xi(x_{0\min s}))/(x-x_{0sp})}\frac{d\nu e^{-i\bar{k}_{1xs}(x-x_{0sp})\nu}}{\left(\nu^2-1\right)^{1/2}}\right\}$$

$$\approx \frac{\sin\delta k_{1x}x_{0sp}/2}{\delta k_{1x}x_{0sp}/2} e^{i\bar{k}_{1xs}x}\left\{2\int_1^{\infty}\frac{d\nu e^{-i\bar{k}_{1xs}(x-x_{0sp})\nu}}{\left(\nu^2-1\right)^{1/2}}-\int_{(x-\xi(x_{0\min s}))/(x-x_{0sp})}^{\infty}\frac{d\nu e^{-i\bar{k}_{1xs}(x-x_{0sp})\nu}}{\left(\nu^2-1\right)^{1/2}}\right\}$$

The first integral is related to a Hankel function, see Gradshteyn and Rhzhic [5] (page 956 #6), so we can write

$$\int_{-\infty}^{\xi(x_{0\min s})} \frac{x_{0sp}-x_0}{\left|x_{0sp}-x_0\right|} \frac{d\xi}{\left(\left(\xi-x\right)^2+2\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}} e^{i\bar{k}_{1xs}\xi}\left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}\right)$$

$$\approx \frac{\sin\delta k_{1x}x_{0sp}/2}{\delta k_{1x}x_{0sp}/2}\left(\frac{\pi}{i2}e^{i\bar{k}_{1xs}x}\right)\left\{2H_0^{(2)}\left(\bar{k}_{1xs}\left(x-x_{0sp}\right)\right)-H_0^{(2)}\left(\frac{x-\xi\left(x_{0\min s}\right)}{x-x_{0sp}},\bar{k}_{1xs}\left(x-x_{0sp}\right)\right)\right\}$$

where the second term in the curly brackets represents an incomplete Hankel function defined by setting the lower limit of the integration to some positive real value that is greater than 1. The contribution to the complex probability density from the blocked trajectories in this case is

$$\rho_1\left(\boldsymbol{x},t;t_0,\boldsymbol{v}_0\right)\approx -i(1/2\pi\hbar^2)N_xN_y\sum_s\frac{e^{(i/\hbar)\left(L_0(t-t_0)-\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}}A_{BS}\left(x_{0sp}\right)V(\bar{k}_{1xs})$$

$$A_{BS}\left(x_{0sp}\right)=\frac{\sin\delta k_{1x}x_{0sp}/2}{x_{0sp}/2}\left(\frac{\pi}{i2}e^{i\bar{k}_{1xs}x}\right)\left\{2H_0^{(2)}\left(\bar{k}_{1xs}\left(x-x_{0sp}\right)\right)-H_0^{(2)}\left(\frac{x-\xi\left(x_{0\min s}\right)}{x-x_{0sp}},\bar{k}_{1xs}\left(x-x_{0sp}\right)\right)\right\}$$

$$x_{0sp}\le x_{0\min s}$$

where the coefficient A_{Bs} has been introduced for comparison with the effects of the unblocked trajectories. When, on the other hand, x_{0sp} is greater than $x_{1\text{mins}}$ we will have

$$\int_{-\infty}^{\xi(x_{0\min s})} \frac{x_{0sp}-x_0}{\left|x_{0sp}-x_0\right|} \frac{d\xi}{\left(\left(\xi-x\right)^2+2\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}} e^{i\bar{k}_{1xs}\xi}\left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}\right)$$

$$= \int_{-\infty}^{\xi(x_{0\min s})} \frac{d\xi}{\left(\left(\xi-x\right)^2+2\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}} e^{i\bar{k}_{1xs}\xi}\left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}\right)$$

the upper limit of the integration is reached before the phase term k_{1x}-bar ξ becomes stationary with respect to x_0. However, it passes through zero before the upper limit is reached, so the sine of $\delta k_{1x}\xi/2$ divided by $\delta k_{1x}\xi/2$ is stationary at 2 points straddling the origin and varies rather slowly between them. Then because the phase term k_{1x}-bar ξ oscillates increasingly rapidly with x_0 as it recedes from the origin,

$$\int_{-\infty}^{\xi(x_{0\min s})} \frac{x_{0sp}-x_0}{\left|x_{0sp}-x_0\right|} \frac{d\xi}{\left(\left(\xi-x\right)^2+2\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}} e^{i\bar{k}_{1xs}\xi}\left(\frac{\sin\delta k_{1x}\xi/2}{\delta k_{1x}\xi/2}\right)$$

$$\approx \frac{\sin\delta k_{1x}\xi\left(x_{0\min s}\right)/2}{\delta k_{1x}\xi\left(x_{0\min s}\right)/2} \int_{-\infty}^{\xi(x_{0\min s})} \frac{d\xi}{\left(\left(\xi-x\right)^2+2\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)}\right)^{1/2}} e^{i\bar{k}_{1xs}\xi}$$

$$\approx \frac{\sin\delta k_{1x}\xi\left(x_{0\min s}\right)/2}{\delta k_{1x}\xi\left(x_{0\min s}\right)/2} \int_{\left(x-\xi\left(x_{0\min s}\right)\right)/\left(x-x_{0sp}\right)}^{\infty} \frac{d\nu}{\left(\nu^2-1\right)^{1/2}} e^{i\bar{k}_{1xs}\left(x-\left(x-x_{0sp}\right)\right)}$$

$$\approx \frac{\sin\delta k_{1x}\xi\left(x_{0\min s}\right)/2}{\delta k_{1x}\xi\left(x_{0\min s}\right)/2}\left(\frac{\pi}{i2}e^{i\bar{k}_{1xs}x}\right)H_0^{(2)}\left(\frac{x-\xi\left(x_{0\min s}\right)}{x-x_{0sp}}, \bar{k}_{1xs}\left(x-x_{0sp}\right)\right)$$

Of course, $\xi(x_{0\mathrm{mins}})$ is less than $x_{0\mathrm{sp}}$ and reaches a maximum at $x_{0\mathrm{sp}}$ as long as the latter is less than x. The coefficient in this case is therefore

$$A_{BS} = \frac{\sin\delta k_{1x}\xi\left(x_{0\min s}\right)/2}{\xi\left(x_{0\min s}\right)/2}\left(\frac{\pi}{i2}e^{i\bar{k}_{1xs}x}\right)H_0^{(2)}\left(\frac{x-\xi\left(x_{0\min s}\right)}{x-x_{0sp}}, \bar{k}_{1xs}\left(x-x_{0sp}\right)\right)$$

2.6.2 Integrating Over the Beam on Unblocked Trajectories

For unblocked trajectories, assuming the quadratic term in the phase can be neglected, we have

$\rho_1(\boldsymbol{x},t;t_0,\boldsymbol{v}_0)$

$$\approx -i(1/2\pi\hbar^2)N_xN_y\sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)\boldsymbol{p}_0\cdot\boldsymbol{x}_0+mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}}\frac{V_0}{\pi}\int_{x_{0\min s}}^{x_{0\max s}}\frac{dx}{a_x}\left\{\begin{array}{l}\dfrac{\pi}{2}\left\{\begin{array}{l}-\dfrac{L}{2}-W<\xi<-\dfrac{L}{2}\\ \dfrac{L}{2}<\xi<\dfrac{L}{2}+W\end{array}\right\}\\ \dfrac{i}{4}\ln\left(\dfrac{\left(L/2+W+\xi\right)\left(L/2-\xi\right)}{\left(L/2+W-\xi\right)\left(L/2+\xi\right)}\right)^2\\ -\displaystyle\int_0^{k_{1x\min}(x_0,s)}\frac{dk_{1x}}{k_{1x}}2\cos k_{1x}\frac{L+W}{2}\sin k_{1x}\frac{W}{2}e^{ik_{1x}\xi}\end{array}\right\}$$

The first two terms will not contribute unless ξ, which is a function of x_0, takes a value near one of the two ranges defining the slits for values of x_0 also in one of the ranges. Since x_0 is (probably chosen to be) small compared to x in such ranges, we can use a linear approximation for ξ

$$\begin{aligned}\xi &= b_x+\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)a_x}\\ &\approx\left(\frac{x}{2}+\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)x}\right)\left(1+x_0/x\right)\equiv\xi_0\left(1+x_0/x\right)\end{aligned}$$

Assuming x is about an order of magnitude larger than x_0 in the relevant ranges, we see that ξ will not vary much during the integration over dx_0, but in order to be in one of the allowed ranges at all, it must be that $\boldsymbol{a}\cdot\boldsymbol{v}_0$ satisfies

$$\frac{\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0}{x}=-\left(1-v_0^2/c^2\right)\frac{x-2\xi_0}{2\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0}$$

for some ξ_0 such that $\xi_0(1+x_0/x)$ and x_0 each fall within the ranges defining one of the two slits. Thus for the terms under consideration to actually contribute, we see that $\boldsymbol{a}\cdot\boldsymbol{v}_0/v_0$ must be negative and about as small compared to x as x is compared to $2\boldsymbol{b}\cdot\boldsymbol{v}_0/v_0$. If that is the case, we will have

$$\int_{x_{0\min s}}^{x_{0\max s}}\frac{dx_0}{a_x}\frac{\pi}{2}\left\{\begin{array}{l}-\dfrac{L}{2}-W<\xi<-\dfrac{L}{2}\\ \dfrac{L}{2}<\xi<\dfrac{L}{2}+W\end{array}\right\}\approx\frac{\pi W}{2x\left(1-\overline{x}_{0s}/x\right)}\left\{\begin{array}{l}-\dfrac{L}{2}-W<\xi_0<-\dfrac{L}{2}\\ \dfrac{L}{2}<\xi_0<\dfrac{L}{2}+W\end{array}\right\}$$

provided $\xi_0(1+x_0/x)$ remains within one of the slits while x_0 varies over the range ($x_{0\text{mins}}$, $x_{0\text{maxs}}$). Otherwise, the result will be smaller going to zero if $\xi_0(1+x_0/x)$ remains outside both slits over the full range.

The second term contributes two integrals of the form

$$\int dx \ln\left(a^2-(bx+c)^2\right)^2 = \frac{2}{b}\left\{\begin{array}{l}\left(a+(bx+c)\right)\ln\left|a+(bx+c)\right|-\left(a+(bx+c)\right)\\ -\left(a-(bx+c)\right)\ln\left|a-(bx+c)\right|+\left(a-(bx+c)\right)\end{array}\right\}$$

which shows that the logarithmic singularities of the integrand will be canceled. Using $b=\xi_0/x$, we can write

$$\frac{i}{4}\int_{x_{0\min s}}^{x_{0\max s}}\frac{dx_0}{a_x}\ln\left(\frac{\left(\frac{L+W}{2}\right)^2-\left(\xi+\frac{W}{2}\right)^2}{\left(\frac{L+W}{2}\right)^2-\left(\xi-\frac{W}{2}\right)^2}\right)^2 \equiv iI_s\left(L,W,x,\xi_0\right)$$

$$\approx \frac{i\left(\frac{L+W}{2}\right)}{2\xi_0\left(1-\bar{x}_{0s}/x\right)}\left[\begin{array}{l}\left(1+\left(\xi+\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right)\ln\left|1+\left(\xi+\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right|-\left(1+\left(\xi+\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right)\\ -\left(1-\left(\xi+\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right)\ln\left|1-\left(\xi+\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right|+\left(1-\left(\xi+\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right)\\ -\left(1+\left(\xi-\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right)\ln\left|1+\left(\xi-\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right|+\left(1+\left(\xi-\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right)\\ +\left(1-\left(\xi-\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right)\ln\left|1-\left(\xi-\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right|-\left(1-\left(\xi-\frac{W}{2}\right)/\left(\frac{L+W}{2}\right)\right)\end{array}\right]_{x_{0\min s}}^{x_{0\max s}}$$

This still peaks near values of ξ corresponding to the edges of the slits as can be seen in Figure 4, which assumes the slits have equal line and space geometry and $(L+W)/2$ equals $0.1x$. Of course, these peaks would be smoothed down if the quadratic term in the phase had not been set to zero. Moreover, the contributions from the upper and lower slits approximately cancel except near their edges.

For small values of ξ, it is useful to rewrite the third term in the form

$$\int_{x_{0\min s}}^{x_{0\max s}}\frac{dx_0}{a_x}\int_0^{k_{1x\min}(x_0,s)}\frac{dk_{1x}}{k_{1x}}2\cos k_{1x}\frac{L+W}{2}\sin k_{1x}\frac{W}{2}e^{ik_{1x}\xi} \approx \int_{x_{0\min s}}^{x_{0\max s}}dx_0\frac{W}{a_x}\int_0^{k_{1x\min}(x_0,s)}dk_{1x}\cos k_{1x}\frac{L+W}{2}e^{ik_{1x}\xi}$$

which is appropriate in the limit that $k_{1\text{xmins}}W/2$ is small compared to unity. Actually, since we are interested in the first minimum at the target, $k_{1\text{xmins}}(L+W)/2$ will be about $\pi/2$. Then, even in the case that the width of the slits is equal to the space between them, $k_{1\text{xmins}}W/2$ would be only $\pi/4$, where the sine is still approximately linear in its argument.

Neglecting the small dependence of $k_{1\text{xmins}}$ on x_0, one can interchange the order of the integrations to get

$$
\begin{aligned}
&\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} dk_{1x} W \cos k_{1x} \frac{L+W}{2} e^{ik_{1x}\xi_0(1+x_0/x)} \\
&\approx \int_0^{k_{1x\min s}} dk_{1x} \left(\cos k_{1x} \frac{L+W}{2} \right) e^{ik_{1x}\xi_0} W \int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} e^{ik_{1x}\xi_0(x_0/x)} \\
&\approx \int_0^{k_{1x\min s}} dk_{1x} \left(\cos k_{1x} \frac{L+W}{2} \right) \frac{e^{ik_{1x}\xi_0} W}{x(1-\bar{x}_{0s}/x)} \left[\frac{e^{ik_{1x}\xi_0(x_0/x)}}{ik_{1x}\xi_0/x} \right]_{x_{0\min s}}^{x_{0\max s}} \\
&\approx \frac{W^2}{x(1-\bar{x}_{0s}/x)} \int_0^{k_{1x\min s}} dk_{1x} \left(\cos k_{1x} \frac{L+W}{2} \right) e^{ik_{1x}\xi_0(1+\bar{x}_{0s}/x)} \frac{\sin k_{1x}\xi_0 (W/2x)}{k_{1x}\xi_0 (W/2x)}
\end{aligned}
$$

Even if $|\xi_0|$ is as large as x, $k_{1x}\xi_0\, W/2x$ will be small in the current assumptions about $k_{1x\min}$ and W, so the integration over k_{1x} can be approximated by

$$
\begin{aligned}
&\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} dk_{1x} W \left(\cos k_{1x} \frac{L+W}{2} \right) e^{ik_{1x}\xi_0(1+x_0/x)} \\
&\approx \frac{W^2}{x(1-\bar{x}_{0s}/x)} \int_0^{k_{1x\min s}} dk_{1x} \left(\cos k_{1x} \frac{L+W}{2} \right) e^{ik_{1x}\xi_0(1+\bar{x}_{0s}/x)} \\
&\approx \frac{W^2}{x(1-\bar{x}_{0s}/x)} \left[\frac{e^{ik_{1x}\left\{\frac{L+W}{2}+\xi_0(1+\bar{x}_{0s}/x)\right\}}}{i2\left\{\frac{L+W}{2}+\xi_0(1+\bar{x}_{0s}/x)\right\}} + \frac{e^{ik_{1x}\left\{-\frac{L+W}{2}+\xi_0(1+\bar{x}_{0s}/x)\right\}}}{i2\left\{-\frac{L+W}{2}+\xi_0(1+\bar{x}_{0s}/x)\right\}} \right]_0^{k_{1x\min s}} \\
&\approx \frac{W^2}{i2x(1-\bar{x}_{0s}/x)} \left[\frac{e^{ik_{1x\min s}\left\{\frac{L+W}{2}+\xi_0(1+\bar{x}_{0s}/x)\right\}}-1}{\frac{L+W}{2}+\xi_0(1+\bar{x}_{0s}/x)} + \frac{e^{ik_{1x\min s}\left\{-\frac{L+W}{2}+\xi_0(1+\bar{x}_{0s}/x)\right\}}-1}{-\frac{L+W}{2}+\xi_0(1+\bar{x}_{0s}/x)} \right]
\end{aligned}
$$

This form is well behaved at small values of ξ. Also, it is almost the same for both slits, since x_{0s}-bar is always divided by x. Even when ξ is at the center of one of the slits $\pm(L+W)/2$, and $k_{1x}(L+W)/2=\pi/2$, we will have

$$
\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} dk_{1x} W \cos k_{1x} \frac{L+W}{2} e^{ik_{1x}\xi_0(1+x_0/x)} \approx \frac{W}{x(1-\bar{x}_{0s}/x)} \left[k_{1x\min} \frac{W}{2} \pm i \frac{W}{L+W} \right]
$$

At these particular values of ξ, we find the real and imaginary parts of the third term partially cancel the first and second terms, respectively.

When $|\xi_0 \pm (L+W)/2|$ are both large compared to $W/2$, it is useful to rewrite the third term in the form

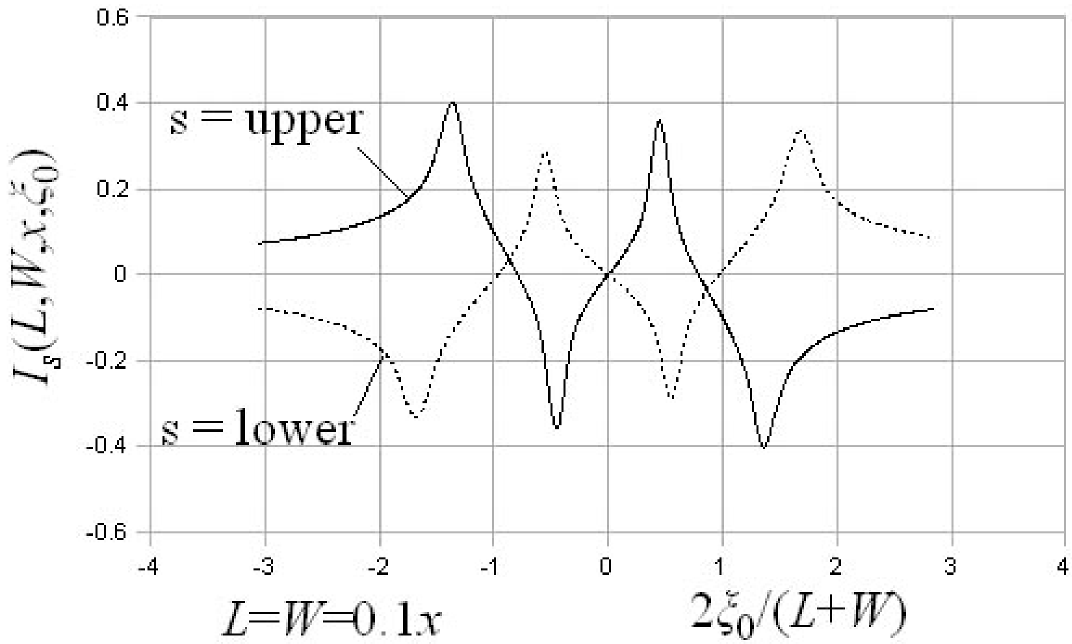

Figure 4 Imaginary part of the infinite integral over momentum transfer

$$\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} \frac{dk_{1x}}{k_{1x}} 2\cos k_{1x}\frac{L+W}{2}\sin k_{1x}\frac{W}{2} e^{ik_{1x}\xi}$$

$$= \int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} \frac{dk_{1x}}{k_{1x}} \left\{ \sin k_{1x}\left(\frac{L+W}{2}+\frac{W}{2}\right) - \sin k_{1x}\left(\frac{L+W}{2}-\frac{W}{2}\right) \right\} e^{ik_{1x}\xi}$$

$$= \int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} \frac{dk_{1x}}{i2k_{1x}} \left\{ e^{ik_{1x}\left(\xi+\frac{L+W}{2}+\frac{W}{2}\right)} - e^{ik_{1x}\left(\xi-\frac{L+W}{2}-\frac{W}{2}\right)} - e^{ik_{1x}\left(\xi+\frac{L+W}{2}-\frac{W}{2}\right)} + e^{ik_{1x}\left(\xi-\frac{L+W}{2}+\frac{W}{2}\right)} \right\}$$

$$= \int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \left\{ \int_0^{k_{1x\min}(x_0,s)\left(\xi+\frac{L+W}{2}+\frac{W}{2}\right)} \frac{du}{i2u}e^{iu} - \int_0^{k_{1x\min}(x_0,s)\left(\xi-\frac{L+W}{2}-\frac{W}{2}\right)} \frac{du}{i2u}e^{iu} - \int_0^{k_{1x\min}(x_0,s)\left(\xi+\frac{L+W}{2}-\frac{W}{2}\right)} \frac{du}{i2u}e^{iu} + \int_0^{k_{1x\min}(x_0,s)\left(\xi-\frac{L+W}{2}+\frac{W}{2}\right)} \frac{du}{i2u}e^{iu} \right\}$$

Then if ξ falls outside both slits

$$\left|\xi \pm \frac{L+W}{2}\right| > \frac{W}{2}$$

the integrals combine in pairs to give

$$\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} \frac{dk_{1x}}{k_{1x}} 2\cos k_{1x} \frac{L+W}{2} \sin k_{1x} \frac{W}{2} e^{ik_{1x}\xi} = \int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \left\{ \int_{k_{1x\min}(x_0,s)\left(\xi+\frac{L+W}{2}-\frac{W}{2}\right)}^{k_{1x\min}(x_0,s)\left(\xi+\frac{L+W}{2}+\frac{W}{2}\right)} \frac{du}{i2u} e^{iu} + \int_{k_{1x\min}(x_0,s)\left(\xi-\frac{L+W}{2}-\frac{W}{2}\right)}^{k_{1x\min}(x_0,s)\left(\xi-\frac{L+W}{2}+\frac{W}{2}\right)} \frac{du}{i2u} e^{iu} \right\}$$

Furthermore, in the limit

$$\left|\xi \pm \frac{L+W}{2}\right| \gg \frac{W}{2}$$

the variation in the denominator becomes negligible over the ranges of the integrations giving

$$\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} \frac{dk_{1x}}{k_{1x}} 2\cos k_{1x} \frac{L+W}{2} \sin k_{1x} \frac{W}{2} e^{ik_{1x}\xi}$$
$$\approx \int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \left\{ \frac{1}{i2k_{1x\min}(x_0,s)\left(\xi+\frac{L+W}{2}\right)} \int_{k_{1x\min}(x_0,s)\left(\xi+\frac{L+W}{2}-\frac{W}{2}\right)}^{k_{1x\min}(x_0,s)\left(\xi+\frac{L+W}{2}+\frac{W}{2}\right)} du e^{iu} + \frac{1}{i2k_{1x\min}(x_0,s)\left(\xi-\frac{L+W}{2}\right)} \int_{k_{1x\min}(x_0,s)\left(\xi-\frac{L+W}{2}-\frac{W}{2}\right)}^{k_{1x\min}(x_0,s)\left(\xi-\frac{L+W}{2}+\frac{W}{2}\right)} du e^{iu} \right\}$$

Neglecting also the small variation of $k_{1x\min}(x_0,s)$ on x_0 and collecting terms gives

$$\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \int_0^{k_{1x\min}(x_0,s)} \frac{dk_{1x}}{k_{1x}} 2\cos k_{1x} \frac{L+W}{2} \sin k_{1x} \frac{W}{2} e^{ik_{1x}\xi}$$
$$\approx \int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \left\{ \frac{e^{ik_{1x\min s}\xi}}{i2\left(\xi+\frac{L+W}{2}\right)} \int_{L/2}^{L/2+W} dx_0' e^{ik_{1x\min s}x_0'} + \frac{e^{ik_{1x\min s}\xi}}{i2\left(\xi-\frac{L+W}{2}\right)} \int_{-L/2-W}^{-L/2} dx_0' e^{ik_{1x\min s}x_0'} \right\}$$
$$\approx \int_{x_{0\min s}}^{x_{0\max s}} \frac{dx_0}{a_x} \frac{e^{ik_{1x\min s}\xi}}{i2\left(\xi^2-\left(\frac{L+W}{2}\right)^2\right)} \left\{ \xi \int_{slits} dx_0' e^{ik_{1x\min s}x_0'} - \frac{L+W}{2}\left[\int_{L/2}^{L/2+W} dx_0' e^{ik_{1x\min s}x_0'} - \int_{-L/2-W}^{-L/2} dx_0' e^{ik_{1x\min s}x_0'} \right] \right\}$$

We have seen that the first two terms in the complex probability due to an unblocked trajectory are only significant when ξ_0 is in the vicinity of the slits. Thus, as the magnitude of ξ_0 increases so that $|\xi_0 \pm (L+W)/2|$ both become large compared to $W/2$, the contribution from unblocked trajectories becomes

$$\rho_1(\boldsymbol{x},t;t_0,\boldsymbol{v}_0) \approx -i(1/2\pi\hbar^2)N_x N_y \sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)\,\boldsymbol{p}_0\cdot\boldsymbol{x}_0 + mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}}$$

$$\times \frac{V_0}{\pi}\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx}{a_x} \frac{ie^{ik_{1x\min s}\xi}}{2\left(\xi^2-\left(\frac{L+W}{2}\right)^2\right)} \left\{-\xi\int_{slits} dx_0' e^{ik_{1x\min s}x_0'} + iW\left(L+W\right)\sin k_{1x\min s}\left(\frac{L+W}{2}\right)\frac{\sin k_{1x\min s}W/2}{k_{1x\min s}W/2}\right\}$$

$$\approx -i(1/2\pi\hbar^2)N_x N_y \sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)\,\boldsymbol{p}_0\cdot\boldsymbol{x}_0 + mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}}$$

$$\times\left\{\begin{array}{l} -\left(\displaystyle\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx}{a_x}\frac{i\xi e^{ik_{1x\min s}\xi}}{\left(\xi^2-\left(\frac{L+W}{2}\right)^2\right)}\right)V\left(k_{1x\min s}\right) \\ -\left(\displaystyle\int_{x_{0\min s}}^{x_{0\max s}} \frac{dx}{a_x}\frac{We^{ik_{1x\min s}\xi}}{2\left(\xi^2-\left(\frac{L+W}{2}\right)^2\right)}\right)\frac{V_0\left(L+W\right)}{\pi}\left(\sin k_{1x\min s}\frac{L+W}{2}\right)\frac{\sin k_{1x\min s}W/2}{k_{1x\min s}W/2} \end{array}\right\}$$

Except for the exponential term, the integrands are almost constant over the small range of values of x_0 in both cases. The average value of ξ in a particular slit s is

$$\begin{aligned}\overline{\xi}_s &= \frac{1}{W}\int_{x_{0\min s}}^{x_{0\max s}} dx_0 \left\{b_x + \frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{\left(1-v_0^2/c^2\right)a_x}\right\} \\ &\approx \frac{1}{W}\int_{x_{0\min s}}^{x_{0\max s}} dx_0\xi_0\left(1+x_0/x\right) \\ &\approx \xi_0\left(1+\frac{x_{0\max s}+x_{0\min s}}{2x}\right) = \xi_0\left(1+\frac{\overline{x}_{0s}}{x}\right)\end{aligned}$$

Using this notation, the approximate result can finally be written

$$\rho_1(\boldsymbol{x},t;t_0,\boldsymbol{v}_0)$$

$$\approx -i(1/2\pi\hbar^2)N_x N_y \sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)\,\boldsymbol{p}_0\cdot\boldsymbol{x}_0 + mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}}$$

$$\times\left\{\begin{array}{l} -\dfrac{ie^{ik_{1x\min s}\bar{\xi}_s}}{x\left(1-\dfrac{\bar{x}_{0s}}{x}\right)}\left(\dfrac{W\bar{\xi}_s}{\bar{\xi}_s^{\,2}-\left(\dfrac{L+W}{2}\right)^2}\right)\left(\dfrac{\sin k_{1x\min s}\xi_0 W/2x}{k_{1x\min s}\xi_0 W/2x}\right)V\left(k_{1x\min s}\right) \\ -\dfrac{e^{ik_{1x\min s}\bar{\xi}_s}}{x\left(1-\dfrac{\bar{x}_{0s}}{x}\right)}\left(\dfrac{W\left(L+W\right)/2}{\bar{\xi}_s^{\,2}-\left(\dfrac{L+W}{2}\right)^2}\right)\left(\dfrac{\sin k_{1x\min s}\xi_0 W/2x}{k_{1x\min s}\xi_0 W/2x}\right)\dfrac{V_0}{\pi}\left(\sin k_{1x\min s}\dfrac{L+W}{2}\right)\dfrac{\sin k_{1x\min s}W/2}{k_{1x\min s}/2} \end{array}\right\}$$

For comparison of the two terms it is useful to isolate the combination of factors and define the complement of the Fourier transform V^C by

$$\begin{aligned} V^C\left(k_{1x\min s}\right) &\equiv \frac{V_0}{\pi}\left(\sin k_{1x\min s}\frac{L+W}{2}\right)\frac{\sin k_{1x\min s}W/2}{k_{1x\min s}/2} \\ &= -\frac{V_0}{k_{1x\min s}\pi}\left(\cos k_{1x\min s}\left(L/2+W\right)-\cos k_{1x\min s}\left(L/2\right)\right) \end{aligned}$$

which has the same amplitude as the Fourier transform of the potential but, being constructed of cosine terms instead of sine terms, is substantially out of phase with it. Then introducing A_U and B_U, as the coefficients of the Fourier transform and its complement in the contributions from the unblocked trajectories, we can write

$$\rho_1(\boldsymbol{x},t;t_0,\boldsymbol{v}_0)$$

$$\approx -i(1/2\pi\hbar^2)N_x N_y \sum_s \frac{e^{(i/\hbar)\left(L_0(t-t_0)\,\boldsymbol{p}_0\cdot\boldsymbol{x}_0 + mc\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{v}_0/c}{\left(1-v_0^2/c^2\right)^{1/2}}\right)}}{\left(1-v_0^2/c^2\right)^{3/2}}\left\{A_{Us}\left(\xi\right)V\left(k_{1x\min s}\right)+B_{Us}\left(\xi\right)\tilde{V}\left(k_{1x\min s}\right)\right\}$$

The ratio of the two coefficients

$$\frac{B_{Us}}{A_{Us}} = i\frac{\left(L+W\right)}{2\bar{\xi}}$$

is small (in absolute magnitude) to the extent that $|\xi|$ is large in comparison with the distance $(L+W)/2$ from the axis of symmetry to the center positions of the slits.

By the way, ξ_0 is related to $x_{0\mathrm{sp}}$ by

$$\xi_0 \equiv \frac{x}{2} + \frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{(1-v_0^2/c^2)x}$$
$$= \frac{x}{2} - \frac{(x-x_{0sp})^2}{2x}$$

and we can see that if x-x_{0sp}>>$3^{1/2}x$ then ξ_0 will be less than $-x$. This means that if x and $-x_{0sp}$ are both large compared to $\{L+W)/2$, ξ_0 will be also.

2.7 Probability Density of Recorded Measurements

When ξ_0 is negative and large in magnitude compared to $(L+W)/2$, the complex probability contributions from the blocked and unblocked trajectories are both proportional to the Fourier transform of the potential with coefficients

$$A_{Bs}(x_{0sp}) = \frac{\sin \delta k_{1x} x_{0sp}/2}{x_{0sp}/2}\left(\frac{\pi}{i2}e^{i\bar{k}_{1xs}x}\right)2H_0^{(2)}\left(\bar{k}_{1xs}(x-x_{0sp})\right)$$

for the blocked trajectories and

$$A_{Us} = -\frac{ie^{ik_{1x\min s}\bar{\xi}_s}}{x\left(1-\dfrac{\bar{x}_{0s}}{x}\right)}\left(\frac{W\bar{\xi}_s}{\bar{\xi}_s^{\,2}-\left(\dfrac{L+W}{2}\right)^2}\right)\left(\frac{\sin k_{1x\min s}\xi_0 W/2x}{k_{1x\min s}\xi_0 W/2x}\right)$$

for the unblocked trajectories. It will be shown that the contributions of the blocked trajectories dominate when x is chosen to be at the first minimum of the diffraction pattern, that is $k_{1x}(L+W)/2$ is equal to $\pi/2$, and x also happens to be large compared to $(L+W)/2$ which in turn is large compared to W.

Considering the blocked trajectories first, it is noted that the argument of the Hankel function is rather large, so we can use the approximation (Gradshteyn and Rhzhic [5] see 8.451 #4 on Page 962)

$$H_0^{(2)}\left(\bar{k}_{1xs}(x-x_{0sp})\right) \approx \left(\frac{2}{\pi\bar{k}_{1xs}(x-x_{0sp})}\right)^{1/2} e^{-i\bar{k}_{1xs}(x-x_{0sp})-i\pi/4}$$

Therefore, the coefficients for the blocked trajectories become

$$A_{Bs}\left(x_{0sp}\right)=\frac{\sin\delta k_{1x}x_{0sp}/2}{x_{0sp}/2}\left(\frac{\pi}{i2}\right)2\left(\frac{2}{\pi\bar{k}_{1xs}\left(x-x_{0sp}\right)}\right)^{1/2}e^{i\bar{k}_{1xs}x_{0sp}-i\pi/4}$$

$$=\frac{\sin\delta k_{1x}x_{0sp}/2}{x_{0sp}/2}\left(\frac{\pi}{i2}\right)2\left(\frac{2}{\pi\bar{k}_{1x}\left(1-\frac{\bar{x}_{0s}}{x}\right)\left(x-x_{0sp}\right)}\right)^{1/2}e^{i\bar{k}_{1x}\left(1-\frac{\bar{x}_{0s}}{x}\right)x_{0sp}-i\pi/4}$$

where we have used

$$\bar{k}_{1xs}\approx\frac{mv_0x}{\hbar\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0}\left(1-\frac{\bar{x}_{0s}}{x}\right)=\bar{k}_{1x}\left(1-\frac{\bar{x}_{0s}}{x}\right)$$

to define k_{1x}-bar. Neglecting the small slit dependence of the amplitudes, the effect of the phase shift between them becomes

$$\sum_s A_{Bs}\left(x_{0sp}\right)=\frac{\sin\delta k_{1x}x_{0sp}/2}{x_{0sp}/2}\left(\frac{\pi}{i2}\right)2\left(\frac{2}{\pi\bar{k}_{1xs}\left(x-x_{0sp}\right)}\right)^{1/2}e^{i\bar{k}_{1x}x_{0sp}-i\pi/4}\left(e^{i\bar{k}_{1x}(L+W)x_{0sp}/2x}+e^{-i\bar{k}_{1x}(L+W)x_{0sp}/2x}\right)$$

$$=\delta k_{1x}\frac{\sin\delta k_{1x}x_{0sp}/2}{\delta k_{1x}x_{0sp}/2}\left(\frac{\pi}{i2}\right)2\left(\frac{2}{\pi\bar{k}_{1xs}\left(x-x_{0sp}\right)}\right)^{1/2}e^{i\bar{k}_{1x}x_{0sp}-i\pi/4}2\left(\cos\bar{k}_{1x}\frac{\left(L+W\right)x_{0sp}}{2x}\right)$$

This envelope of this expression decreases slowly as x_{0sp} retreats from $-x$ to larger negative values as long as $\delta k1_x x_{0sp}/2$ remains small

$$\left|\delta k_{1x}x_{0sp}/2\right|\approx k_{1x}\frac{w}{x}\left|x_{0sp}\right|/2$$

$$\approx\left(k_{1x}\frac{L+W}{2}\right)\frac{W}{L+W}\left|\frac{x_{0sp}}{x}\right|<\frac{\pi}{2}$$

In particular, if x is chosen to be the first null in the diffraction pattern, then this condition is equivalent to

$$\left|\frac{x_{0sp}}{x}\right|<\frac{L+W}{W}$$

Thus the range of (negative) values of x_{0sp} in which the sum of the coefficients for the blocked trajectories can be significant is large (compared to x) to the extent $W/(L+W)$ is small. Within this range, the sum of the coefficients for the blocked trajectories becomes

$$\sum_s A_{Bs}\left(x_{0sp}\right)\approx-i\delta k_{1x}\left(\frac{2\pi}{\bar{k}_{1xs}\left(x-x_{0sp}\right)}\right)^{1/2}e^{i\bar{k}_{1x}x_{0sp}-i\pi/4}\left(2\cos\bar{k}_{1x}\frac{\left(L+W\right)x_{0sp}}{2x}\right)$$

Turning now to the blocked trajectories, there is also a phase shift between the contributions from the two slits. Actually, both terms in the phase vary and

$$\begin{aligned}
\bar{k}_{1x\min s}\bar{\xi}_s &\approx \bar{k}_{1x}\left(1-\frac{x_{0\max s}}{x}\right)\bar{\xi}_s \\
&= \bar{k}_{1x}\xi_0\left(1-\frac{x_{0\max s}}{x}\right)\left(1+\frac{\bar{x}_{0s}}{x}\right) \\
&= \bar{k}_{1x}\xi_0\left(1-\frac{\bar{x}_{0s}}{x}-\frac{W}{2x}\right)\left(1+\frac{\bar{x}_{0s}}{x}\right) \\
&\approx \bar{k}_{1x}\xi_0\left(\left(1-\frac{W}{2x}\right)-\frac{W}{2x}\frac{\bar{x}_{0s}}{x}-\left(\frac{\bar{x}_{0s}}{x}\right)^2\right)
\end{aligned}$$

Averaging the small difference in the magnitudes of the two terms, the net contribution from the two slits due to unblocked trajectories is given (to lowest order in W/x) is given by

$$\sum_s A_{Us} \approx -\frac{i}{x}\left(\frac{W\xi_0}{\xi_0^{\;2}-\left(\frac{L+W}{2}\right)^2}\right)\left(\frac{\sin\bar{k}_{1x}\xi_0 W/2x}{\bar{k}_{1x}\xi_0 W/2x}\right)e^{i\bar{k}_{1x}\xi_0\left(1-\frac{W}{2x}-\left(\frac{L+W}{2x}\right)^2\right)}2\cos\bar{k}_{1x}\xi_0\frac{W}{2x}\frac{L+W}{2x}$$

The envelope of this expression decreases like $(\xi_0)^{-1}$ as ξ_0 retreats from $-x$ towards more negative values as long as k_{1x}-bar $\xi_0 W/2x$ remains small

$$\bar{k}_{1x}\xi_0 W/2x = \left(\bar{k}_{1x}\frac{L+W}{2}\right)\frac{W}{L+W}\frac{\xi_0}{x}$$

From its definition, we can see that ξ_0

$$\begin{aligned}
\xi_0 &= \frac{x}{2}+\frac{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0)(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0)}{(1-v_0^2/c^2)x} \\
&= \frac{x}{2}-\frac{(x-x_{0sp})^2}{2x} \\
&= x_{0sp}-\frac{x_{0sp}^2}{2x}
\end{aligned}$$

is approximately equal to x_{0sp} when $|x_{0sp}|$ is small compared to x. As long as $|\xi_0|$ is large compared to $(L+W)/2$, but less than about $(1+L/W)x$, the net contribution from the two slits due to the unblocked trajectories will be approximately

$$\sum_s A_{Us} \approx -\frac{i}{x}\left(\frac{W\xi_0}{\xi_0^{\ 2} - \left(\frac{L+W}{2}\right)^2}\right) e^{i\bar{k}_{1x}\xi_0\left(1-\frac{W}{2x}-\left(\frac{L+W}{2x}\right)^2\right)} 2\cos\bar{k}_{1x}\xi_0\frac{W}{2x}\frac{L+W}{2x}$$

At larger values of $|x_{0sp}|$ on the order of x, ξ_0 begins to grow quadratically. This means that as x_{0sp} retreats from $-x$ towards larger negative values, the contribution from the unblocked trajectories will cut off before that of the blocked trajectories does. For example, when x-x_{0sp} is larger than $3^{1/2}x$, ξ_0 will be less than $-x$. The envelope of the blocked trajectories begins to decrease like $(\xi_0)^{-2} \sim (x_{osp})^{-4}$ when $\xi_0/x < -(L+W)/W$ which occurs when

$$\frac{x_{0sp}}{x} < 1 - \left(1 + 2\frac{L+W}{W}\right)^{1/2}$$

If (L+W)/W is large, the range in x0sp in which the blocked trajectory contributions remain significant after the unblocked trajectories have cut off

$$-\left(1+\frac{L}{W}\right)x < x_{0sp} < \left(1 - \left(1 + 2\left(1+\frac{L}{W}\right)\right)^{1/2}\right)x$$

Even before unblocked trajectories cut off, for example where ξ_0=$-x$ the peak values of their amplitude

$$\left|\frac{\sum_s A_{Us}}{\sum_s A_{Bs}\left(x_{0sp}\right)}\right| \approx \frac{1}{\delta k_{1x} x}\left(\frac{W|\xi_0|}{\xi_0^{\ 2} - \left(\frac{L+W}{2}\right)^2}\right)\left(\frac{\bar{k}_{1x}\left(x - x_{0sp}\right)}{2\pi}\right)^{1/2}$$

$$\approx \frac{1}{\bar{k}_{1x}}\left(\frac{|\xi_0|}{\xi_0^{\ 2} - \left(\frac{L+W}{2}\right)^2}\right)\left(\frac{\bar{k}_{1x}\left(x - x_{0sp}\right)}{2\pi}\right)^{1/2}$$

$$\approx \left(\frac{|\xi_0 / x|}{\left(\xi_0 / x\right)^2 - \left(\frac{L+W}{2x}\right)^2}\right)\left(\frac{\left(1-\frac{x_{0sp}}{x}\right)\left(\frac{L+W}{2x}\right)}{2\pi\bar{k}_{1x}\left(\frac{L+W}{2}\right)}\right)^{1/2}$$

is small compared to that of the blocked trajectories. That is, at the first minimum of the Fourier transform of the potential $2\pi k_{1x}$-bar$(L+W)/2$ is about π^2. Also, still assuming x is 10 times $(L+W)/2$, the ratio of the amplitudes will be on the order of 10^{-1} times a function of the reduced variable ξ_0/x (or x_{0sp}/x). For example, when x-x_{0sp} is $3^{1/2}x$ giving $\xi_0 = -x$, the function is about $3^{1/4}$ and the blocked trajectories are still dominant.

By hypothesis b), the differential probability of recording the particle at $\boldsymbol{x}$ in time dt when x-x_{0sp} is larger than $3^{1/2}x$ is therefore approximately

$$
\begin{aligned}
&\left|\rho_1\left(\boldsymbol{x},t;t_0,\boldsymbol{v}_0\right)\right|^2 dt \\
&\sim\left(\left(1/2\pi\hbar^2\right)\frac{N_xN_y}{\left(1-v_0^2/c^2\right)^{3/2}}\right)^2\left|\sum_s V(\bar{k}_{1xs})\delta k_{1x}\left(\frac{\sin\delta k_{1x}x_{0sp}/2}{\delta k_{1x}x_{0sp}/2}\right)\left(\frac{\pi}{\bar{k}_{1xs}\left(x-x_{0sp}\right)}\right)^{1/2}\left(\cos\bar{k}_{1x}\frac{\left(L+W\right)x_{0sp}}{2x}\right)\right|^2 dt
\end{aligned}
$$

Next consider the change of variable

$$
\begin{aligned}
x-x_{0sp}&=\left\{-2\frac{\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)\left(\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0\right)}{1-v_0^2/c^2}\right\}^{1/2}\\
&=\left\{-\frac{\left(\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right)^2-v_0^2\left(t-t_0\right)^2}{1-v_0^2/c^2}\right\}^{1/2}
\end{aligned}
$$

$$
-dx_{0sp}=\frac{1}{x-x_{0sp}}\frac{v_0^2\left(t-t_0\right)dt}{1-v_0^2/c^2}
$$

which then leads to

$$
\begin{aligned}
&\left|\rho_1\left(\boldsymbol{x},t;t_0,\boldsymbol{v}_0\right)\right|^2 dt \\
&\sim\left(\left(1/2\pi\hbar^2\right)\frac{N_xN_y}{\left(1-v_0^2/c^2\right)^{3/2}}\right)^2\frac{\left(1-v_0^2/c^2\right)}{v_0^2\left(t-t_0\right)}\frac{\pi\left(\delta k_{1x}\right)^2}{\bar{k}_{1xs}}\left(\frac{\sin\delta k_{1x}x_{0sp}/2}{\delta k_{1x}x_{0sp}/2}\right)^2\left|V(\bar{k}_{1xs})\left(\cos\bar{k}_{1x}\frac{\left(L+W\right)x_{0sp}}{2x}\right)\right|^2 dx_{0sp}
\end{aligned}
$$

The change of integration variable from dt to dx_{0sp} cancels the factor of x-x_{0sp}, leaving a relatively simple function at the lower limit of the integration. The integrand has a zero at $\delta k_{1x}x_{0sp}/2 = -\pi$ and remains small for x_{0sp}/x less than about $-2\pi(L+W)/W$. It also has zeros where x_{0sp}/x is a negative odd integer assuming that the first null in the diffraction pattern is at x. Strictly speaking, the factor of t-t_0 is not constant, but solving for it in terms of x_{0sp}

$$
\begin{aligned}
v_0\left(t-t_0\right)&=\left(\left(\boldsymbol{x}\cdot\hat{\boldsymbol{v}}_0\right)^2+\left(1-v_0^2/c^2\right)\left(x-x_{0sp}\right)^2\right)^{1/2}\\
&=\left(\left|\boldsymbol{x}\right|^2+\left(1-v_0^2/c^2\right)\left(x-x_{0sp}\right)^2-x^2\right)^{1/2}
\end{aligned}
$$

we see that the integrand will become very small before $v_0(t$-$t_0)$ becomes significantly different from $|\boldsymbol{x}|$. It's appearance in the denominator is appropriate for the 2-dimensional experiment being considered.

When x_{0sp} approaches zero from negative values, $k_{1x}(x$-$x_{0sp})$ remains large, so the complex probability contribution from the blocked trajectories approaches a value that has the same functional form on x_{0sp} but is divided by 2. However, we find ξ_0

$$\xi_0 = x_{0sp} - \frac{x_{0sp}^2}{2x}$$

converges to x_{0sp}. This means that the component of the complex probability density due to the unblocked trajectories that is in phase with the Fourier transform of the potential decreases linearly while the out-of-phase component remains substantially constant.

Once x_{0sp} turns positive, though, the contribution from the blocked trajectories will decrease as the lower limit of the incomplete Hankel function increases from unity towards larger values where the integrand is smaller in magnitude. The unbound trajectories may then dominate as x_{0sp} approaches x, which is the upper limit of the range over which the phase is stationary at some value of k_{1x}, ξ_0 approaches $x/2$. At this upper limit, the contribution from the unblocked trajectories will again be essentially proportional to the Fourier transform assuming $x/2$ is large compared to $(L+W)/2$. However, the factor of x-x_{0sp} resulting from changing the integration variable from t to x_{0sp} will then cancel this remaining contribution from the unblocked trajectories.

On the other hand, the range of arrival times at which the contributions from the blocked trajectories can be significant corresponds roughly to

$$-\pi \le \delta k_{1x} x_{0sp} / 2 \le 0$$

Since δk_{1x} is roughly W/x smaller than k_{1x}, this means the lower limit of x_{0sp} is on the order of $-2\pi x/k_{1x}W$ or

$$x_{0sp} \ge -\frac{2\left(L+W\right)x}{W}$$

when the first minimum is at x. Therefore x-x_{0sp} can be large compared to x if W is small compared to $(L+W)/2$ and still as large as $5x$ for W equal to L. Thus it emerge*s* that the blocked trajectories can be significant relatively far from the slits, compared to the observation point, and therefore contribute much more than the unblocked trajectories.

We can now estimate the change in the energy of a particle on a trajectory defined by x_0 arriving at the target position at time t given in terms of x_{0sp}

$$\begin{aligned} T_1 - T_0 &= \frac{\hbar k_{1x} \boldsymbol{a} \cdot \boldsymbol{v}_0}{a_x\left(1-v_0^2/c^2\right)} + \frac{mc^2}{\left(1-v_0^2/c^2\right)^{1/2}} \left\{ \left\{ 1 + \left(a^2 + \frac{\left(\boldsymbol{a}\cdot\boldsymbol{v}_0/c\right)^2}{1-v_0^2/c^2} \right) \left(\frac{\hbar k_{1x}}{mca_x} \right)^2 \right\}^{1/2} - 1 \right\} \\ &\approx \frac{\hbar k_{1x}}{a_x} \left\{ \frac{\boldsymbol{a}\cdot\boldsymbol{v}_0}{\left(1-v_0^2/c^2\right)} + \frac{a^2}{2\left(1-v_0^2/c^2\right)^{1/2}} \left(\frac{\hbar k_{1x}}{ma_x} \right) \right\} \end{aligned}$$

because the component of $\boldsymbol{a}$ along the original direction of motion is related to x_{0sp} by

$$\boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 = -\left(1 - v_0^2/c^2\right)\frac{\left(x - x_{0xp}\right)^2}{2\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0}$$

and is therefore bounded by

$$-\left(1 - v_0^2/c^2\right)\frac{x^2}{2\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0} \geq \boldsymbol{a}\cdot\hat{\boldsymbol{v}}_0 \geq -\left(1 - v_0^2/c^2\right)\frac{x^2}{2\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0}\left(1 + \frac{2\left(L+W\right)}{W}\right)^2$$

Therefore the non-conservation of energy for a particle entering with some offset x_0 becomes

$$\begin{aligned} T_1 - T_0 &\approx \frac{\hbar k_{1x}}{a_x}\left\{\frac{\boldsymbol{a}\cdot\boldsymbol{v}_0}{\left(1 - v_0^2/c^2\right)} + \frac{a^2}{2\left(1 - v_0^2/c^2\right)^{1/2}}\left(\frac{\hbar k_{1x}}{m a_x}\right)\right\} \\ &\approx \hbar k_{1x}\left\{-\frac{\left(x - x_{0sp}\right)^2 v_0}{2\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)a_x} + \frac{a^2/a_x^2}{2\left(1 - v_0^2/c^2\right)^{1/2}}\left(\frac{\hbar k_{1x}}{m}\right)\right\} \end{aligned}$$

The peak contribution at this time comes from particles entering with offset $x_0 = x_{0sp}$. Also, since the component of $\boldsymbol{a}$ along the original direction is small, we can neglect the small difference between a and $a_x = x - x_{0sp}$

$$T_1 - T_0 \approx \hbar k_{1x}\left\{-\frac{\left(x - x_{0sp}\right)v_0}{2\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)} + \frac{1}{2\left(1 - v_0^2/c^2\right)^{1/2}}\left(\frac{\hbar k_{1x}}{m}\right)\right\}$$

Since $\hbar k_{1x}/m$ and xv_0/b are both approximately equal to v_{1x}, the two terms cancel at the upper limit corresponding to $x_{0sp}\approx 0$ giving us (roughly)

$$-\hbar k_{1x} v_0 \frac{x}{2\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0}\left(2\frac{L+W}{W}\right) \leq T_1 - T_0 \leq 0$$

The maximum energy loss is therefore about $2(L+W)/W$ times the kinetic energy that the particle would have after the interaction due to its motion in the x-direction alone. To appreciate how small that is, note that at the first minimum it can be expressed in terms of the slit geometry and particle mass and $\hbar^2$

$$\hbar k_{1x} v_0 \frac{x}{\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0}\left(\frac{L+W}{W}\right) = \pi\hbar v_0 \frac{x}{W\left(\boldsymbol{b}\cdot\hat{\boldsymbol{v}}_0\right)}$$

$$\approx \pi\hbar \frac{v_{1x}}{W}$$

$$\approx \frac{\pi^2\hbar^2}{mW\left(L+W\right)}$$

The fractional energy loss in the non-relativistic limit can also be expressed as

$$-\left(\frac{\hbar k_{1x} 2\left(L+W\right)}{mv_0}\right)^2 \left(\frac{1}{2W\left(L+W\right)}\right) \le \frac{\Delta T}{mv_0^2/2} \le 0$$

provided the scattering angle $x/|\boldsymbol{x}|$ remains small. Of course, the assumption that the scattering angle is small is equivalent to assuming that the de Broglie wavelength is small compared to $L+W$. If x corresponds to the first side peak in the diffraction pattern, that is $k_{1x}(L+W)=2\pi$, the range of the fractional energy loss should be in the range

$$-\frac{2\lambda_{dB}^2}{W\left(L+W\right)} \le \frac{\Delta T}{mv_0^2/2} \le 0$$

which will be largest with light particles at low energy.

The maximum change in energy may be just observable with electrons at low energy. High-resolution electron energy loss (HREEL) spectroscopy [6] is typically carried out with incident energies on the order of 10 electron volts corresponding to de Broglie wavelengths on the order of several Angstroms. Fractional energy shifts on the order of 10^{-3} can be resolved in HREEL. If we assume that the width of the slits and the un-patterned line between them are equal, then a fractional energy shift of 10^{-3} would result when the slit and line widths are about $10^{1.5}$ times the de Broglie wavelength, or on the order of 10 nm. Patterning such fine features in a free-standing film may be slightly beyond the state of the art. However, 100-nm gratings have already been fabricated and used to study the diffraction of electrons with energies down to 125 electron volts [7].

3. Interpretation

The traditional theory of quantum mechanics gives diffraction by treating material particles as waves, leaving one with a conundrum when waves arriving along disjoint trajectories interfere at the detector. In our approach, we also add such contributions, but the diffraction pattern is also more physically related to the Fourier transform of the potential. Even though this potential may actually be zero everywhere along a path through one of the openings in the double-slit experiment, its Fourier components are not necessarily zero. According to our proposed hypothesis, the Fourier components are each independently responsible for interactions.

The interference pattern changes radically when one of the slits is closed, of course, but we can say that this is because the Fourier spectrum of the potential changes. Thus we are not as tempted to say that the interference pattern is caused by our lack of knowledge about the trajectory of the particle. It seems reasonable to assert that the standard method of solving the double-slit experiment obscures interactions with the atoms in the barrier, which Feynman probably would have insisted are ultimately in play. On this point, we note that the sum of the Fourier transforms of a collection of sources such as atoms, must equal the Fourier transform of the net potential created by that collection.

Even from our point of view, however, the form of the interference pattern can be modified by averaging non-positive quantities over all possible trajectories the particle might have taken. We would argue that this remaining violation of intuition may be no more (nor less) mysterious than the change of sign in the wave function of an electron upon rotating the coordinate system by 360 degrees, which is a mathematical property of half-integer spin representations of the group of rotations in three dimensions. In our case, the addition of non-positive probabilities results primarily from a simple implementation of our quantum hypothesis. To our continual surprise, nature seems to make use of all mathematical degrees of freedom. Consequently, we must prepare to be flexible when researching relationships between memories of past events.

The use of physical paths generated by discrete momentum increments entails many paths along which the particle's energy is not conserved. We have explored the significance of this fact in the double-slit experiment. In the single-quantum-transfer approximation, it turns out that energy is exactly conserved when the phase is stationary. However, the phase doesn't vary as rapidly (with position of the quantum transfer on the initial trajectory) as might be expected. Also, we find that the contributions from the trajectories on which the particle scatters *before* reaching the plane of the slits are generally more significant. The phase is not necessarily stationary on such trajectories, but the transfer is geometrically restricted to small ranges from which the target position is visible through one of the slits. In non-relativistic cases, the maximum change in energy is a loss that is proportional to the square of Planck's constant when expressed in terms of the slit geometry and the mass of the particle. It is largest for light particles at low energy and might be observable with electrons at around 10 eV. Of course, the predicted change might be reduced when multiple quantum transfers are included. It could also be reduced by a more realistic treatment of the aperture plane, which we have represented by a static potential energy function.

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